The word problem for certain Hecke–Kiselman monoids

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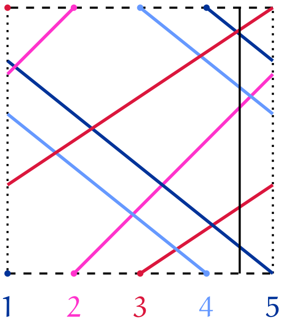


1 Summary

Linear Hecke–Kiselman monoids L_n (of type A_n):

- generators x_i, $1 \leqslant i \leqslant n$;
- relations

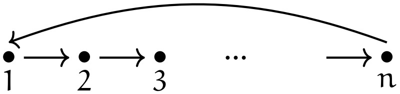
$$
\begin{aligned}
&x_i^2 = x_i, && 1 \leqslant i \leqslant n,\\
&x_i x_j = x_j x_i, && 1 < i - j < n,\\
&x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1} = x_i x_{i+1}, && 1 \leqslant i < n.
\end{aligned}
$$

Part 1: We will see that:

① they are interesting;

② one knows a lot about them.

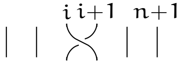
Part 2: Same for circular Hecke–Kiselman monoids C_n (of type $\widetilde{A}_n$).



2 Why these relations?

Positive braid monoids B^+_{n+1}:

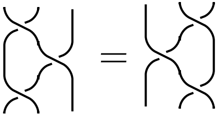
- generators x_i, $1 \leqslant i \leqslant n$

$x_i \leftrightarrow$ 

- relations

$$x_i x_j = x_j x_i, \qquad 1 < i - j < n,$$

$$x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1}, \qquad 1 \leqslant i < n.$$

$x_1 x_2 x_1 = x_2 x_1 x_2 \leftrightarrow$  (Reidemeister III move)

2 Why these relations?

Positive braid monoids B^+_{n+1}:

- generators x_i, $1 \leqslant i \leqslant n$
- relations

$$x_i x_j = x_j x_i, \qquad 1 < i - j < n,$$
$$x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1}, \qquad 1 \leqslant i < n.$$

Finite quotients?

Ⓐ $x_i^2 = 1$: symmetric group S_{n+1};

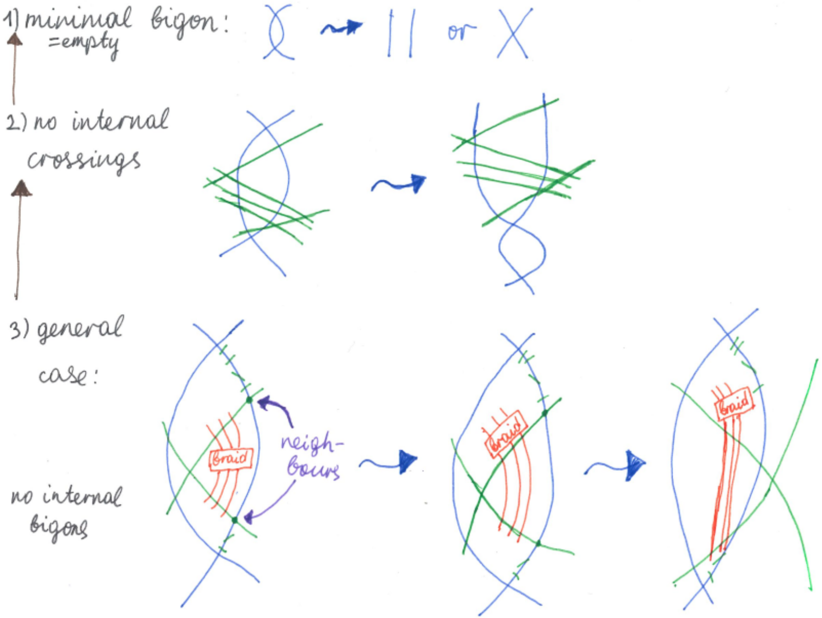
Ⓑ $x_i^2 = x_i$: 0-Hecke monoids;

Ⓒ $x_i^2 = 0$ (with an additional generator 0): nil-Hecke monoids;

Ⓓ monoid algebra + general quadratic relation: Hecke algebra.

Why finite? **Bigon killing!**

Bigon killing



2 Why these relations?

Linear Hecke–Kiselman monoids L_n (*Ganyushkin–Mazorchuk '02*):

- generators x_i, $1 \leqslant i \leqslant n$
- relations

$$
\begin{aligned}
x_i^2 &= x_i, && 1 \leqslant i \leqslant n,\\
x_i x_j &= x_j x_i, && 1 < i - j < n,\\
x_i x_{i+1} x_i &= x_{i+1} x_i x_{i+1} = x_i x_{i+1}, && 1 \leqslant i < n.
\end{aligned}
$$

Positive braid monoids B^+_{n+1} $\rightsquigarrow$ 0-Hecke monoids + Kiselman monoids.

(convexity theory)

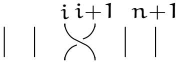
Also appear in:

- computer simulations (discrete sequential dynamical system, *Collina–D'Andrea '15*)

$$
\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet} \quad \cdots \quad \longrightarrow \underset{n}{\bullet}
$$

- representations of path algebras of quivers (projection functors, *Grensing–Mazorchuk, '12-'17*).

3 Yang–Baxter-like representations

$x_i \;\leftrightarrow$ 

Definition: L_n-chain on A = idempotent maps $\sigma_i \colon A^2 \to A^2$ satisfying

$$\begin{aligned}(\sigma_i \times \mathrm{Id})(\mathrm{Id} \times \sigma_{i+1})(\sigma_i \times \mathrm{Id}) &= (\mathrm{Id} \times \sigma_{i+1})(\sigma_i \times \mathrm{Id})(\mathrm{Id} \times \sigma_{i+1}) \\ &= (\sigma_i \times \mathrm{Id})(\mathrm{Id} \times \sigma_{i+1}) \qquad \text{on } A^3.\end{aligned}$$

Proposition: L_n acts on A^{n+1} by

$$x_i \mapsto \mathrm{Id}^{i-1} \times \sigma_i \times \mathrm{Id}^{n-i}.$$

Remark: All $\sigma_i = \sigma \rightsquigarrow$ idempotent Kiselman Yang–Baxter operator.

Examples:

- $\sigma_i(a, b) = (a, p_i(b))$, with $p_i^2 = p_i$.
- $\sigma_i(a, b) = (a, f_i(a))$.

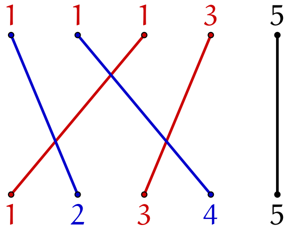
3 Yang–Baxter-like representations

Examples:

- $\sigma_i(a, b) = (a, p_i(b))$, with $p_i^2 = p_i$.
- $\sigma_i(a, b) = (a, f_i(a))$.

  Particular case: $\sigma_i(a, b) = (a, a)$ recovers

$$L_n \quad \overset{1:1}{\leftrightarrow} \quad \mathrm{Cat}_{n+1} \text{ (Catalan monoid)}$$



3 Yang–Baxter-like representations

Examples:

- $\sigma_i(a, b) = (a, p_i(b))$, with $p_i^2 = p_i$.
- $\sigma_i(a, b) = (a, f_i(a))$.
- $\sigma_i(a, b) = (1, f_i(a)b)$, with A a monoid, and f_i monoid homomorphisms.
- $\sigma_i(a, b) = (a, a * b)$, with $*$ associative and absorbing:
$$a * (a * b) = a * b.$$

4 Understanding a monoid

1. size;
2. word problem;
3. normal form.

5 Webs

Theorem (folklore): There are explicit bijections between:

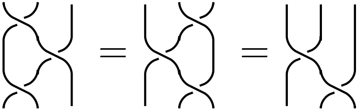
(A) the elements of L_n;

(B) n-webs (weakly entangled braids):
bigons-less and triangle-less positive braids on $n+1$ strands;

Proof idea for (A) → (B):

$x_i^2 = x_i$ bigon killing

$x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1} = x_i x_{i+1}$ triangle killing



Subtlety: different killing schemes.

Corollary: rewriting procedure.

6 Π-sequences

Theorem (folklore): There are explicit bijections between:

(A) the elements of L_n;

(B) bigons-less and triangle-less positive braids on $n+1$ strands;

(C) increasing couples of increasing integer sequences between 1 and $n+1$:

$$\begin{array}{ccccccccccc} & & b_1 & < & b_2 & < & \dots & < & b_k & \leqslant & n+1 \\ & & \vee & & \vee & & \dots & & \vee & & \\ 1 & \leqslant & a_1 & < & a_2 & < & \dots & < & a_k & & \end{array}$$

6 II-sequences

Theorem (folklore): There are explicit bijections between:

Ⓑ bigons-less and triangle-less positive braids on $n+1$ strands;

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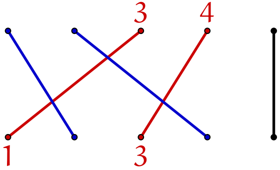
Proof idea for Ⓑ → Ⓒ: follow the right strands.

Example:



Proof idea for Ⓒ → Ⓑ: draw the right strands and complete.

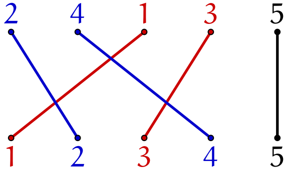
7 Permutations

Theorem (folklore): There are explicit bijections between:

Ⓑ bigons-less and triangle-less positive braids on $n+1$ strands;

Ⓓ 321-avoiding permutations from S_{n+1}.

Example:



8 Answers for L_n

Theorem (folklore): There are explicit bijections between:

Ⓐ the elements of L_n;

Ⓑ bigons-less and triangle-less positive braids on $n+1$ strands;

Ⓒ increasing couples of increasing integer sequences between 1 and $n+1$;

Ⓓ 321-avoiding permutations from S_{n+1}.

Proof idea for Ⓐ $\to$ Ⓒ: use the L_n-chain $\sigma_i(a, b) = (a, a)$.

Corollaries:

1 size: Catalans $C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1}$ (byproduct: their exotic avatars);

2 word problem: a linear solution Ⓐ $\to$ Ⓒ;

3 a quadratic normal form: Ⓐ $\to$ Ⓒ $\to$ Ⓑ $\to$ Ⓐ

or Ⓐ $\to$ Ⓒ $\to$ Ⓓ $\xrightarrow[\text{process}]{\text{inductive}}$ Ⓐ.

9 Big brother

Circular Hecke–Kiselman monoids C_n (of type $\widetilde{A}_n$), $n \geqslant 3$:

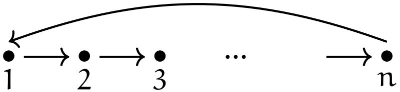
- generators x_i, $1 \leqslant i \leqslant n$;
- relations

$$x_i^2 = x_i, \qquad 1 \leqslant i \leqslant n,$$

$$x_i x_j = x_j x_i, \qquad 1 < i - j < n-1,$$

$$x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1} = x_i x_{i+1}, \qquad 1 \leqslant i < n+1,$$

where x_{n+1} means x_1.



What was known for C_n

1 size: infinite;

2 word problem: two versions of the same solution:

- a finite Gröbner basis (*Męcel–Okniński '19*);
- confluent reductions (*Aragona–D'Andrea '20*);

3 a complicated normal form for almost all elements (*Okniński–Wiertel '20*).

Application: the algebra $K[C_n]$ is Noetherian.

11 Yang–Baxter-like representations

Definition: C_n-chain on A = idempotent maps $\sigma_i : A^2 \to A^2$ satisfying

$$(\sigma_i \times \mathrm{Id})(\mathrm{Id} \times \sigma_{i+1})(\sigma_i \times \mathrm{Id}) = (\mathrm{Id} \times \sigma_{i+1})(\sigma_i \times \mathrm{Id})(\mathrm{Id} \times \sigma_{i+1})$$

$$= (\sigma_i \times \mathrm{Id})(\mathrm{Id} \times \sigma_{i+1}) \qquad \text{on } A^3$$

for $1 \leqslant i \leqslant n$. As usual, we put $\sigma_{n+1} = \sigma_1$.

Proposition: C_n acts on A^n by

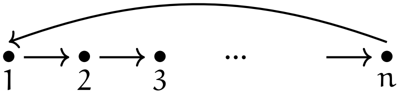
$$x_i \mapsto \mathrm{Id}^{i-1} \times \sigma_i \times \mathrm{Id}^{n-i} \qquad \text{for all } i < n,$$

$$x_n \mapsto \theta^{-1}(\sigma_n \times \mathrm{Id}^{n-2})\theta,$$

where θ is the permutation moving the last component to the beginning.

Examples: The same as for L_n. For instance, $\sigma_i(a, b) = (a, f_i(a))$.

Particular case: $\sigma_i(a, b) = (a, a)$, $i < n$, and $\sigma_n(a, b) = (a, a + 1)$
(*Aragona–D'Andrea '13*).



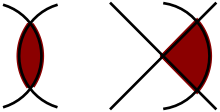
12 Webs on a cylinder

Theorem (*L. '21*): There are explicit bijections between:

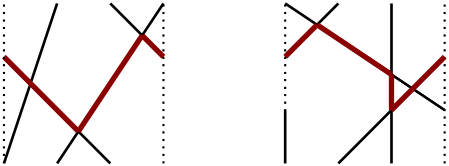
(A) the elements of C_n;

(B) $\tilde{n}$-webs (weakly entangled braids): positive n-strand braids on a cylinder

- without contractible bigons and triangles

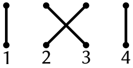


contractible



non-contractible

- and compositions of elementary diagrams:

$d_2 =$ 

12 Webs on a cylinder

Theorem (*L. '21*): There are explicit bijections between:

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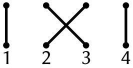
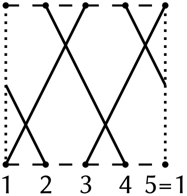
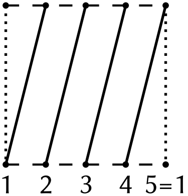
- without contractible bigons and triangles
- and composed from elementary diagrams:

$d_2 =$ 

Examples:



✓

t= 

✗

Remark: The d_i and t generate the braid monoid/group on the cylinder.

12 Webs on a cylinder

Theorem (*L. '21*): There are explicit bijections between:

Ⓐ the elements of $\mathrm{C}_{\mathfrak{n}}$;

Ⓑ $\tilde{\mathfrak{n}}$-webs (weakly entangled braids): positive $\mathfrak{n}$-strand braids on a cylinder

- without contractible bigons and triangles
- and composed from elementary diagrams.

Proof idea for Ⓐ → Ⓑ: Kill all contractible bigons and triangles.

Subtlety: different killing schemes.

Corollary: rewriting procedure.

II-sequences

Theorem (*L. '21*): There are explicit bijections between:

(B) $\tilde{\mathfrak{n}}$-webs on a cylinder;

(C) $\mathfrak{n}$-close increasing couples of increasing integer sequences:

$$\begin{array}{ccccccccccc} & & b_1 & < & b_2 & < & \ldots & < & b_k & < & b_1 + n \\ & & \vee & & \vee & & \ldots & & \vee & & \\ 1 & \leqslant & a_1 & < & a_2 & < & \ldots & < & a_k & \leqslant & n \end{array}$$

Proof idea for (B) $\to$ (C): follow the right strands.

Proposition: For an $\tilde{\mathfrak{n}}$-diagram, the following are equivalent:

1. no contractible bigons, no contractible triangles;
2. no *minimal* contractible bigons, no *minimal* contractible triangles;
3. up to isotopy, each strand is right, left or vertical.

13 II-sequences

Theorem (*L. ’21*): There are explicit bijections between:

Ⓑ $\tilde{n}$-webs (weakly entangled braids) on a cylinder;

Ⓒ n-close increasing couples of increasing integer sequences:

$$\begin{array}{ccccccccccc} & & b_1 & < & b_2 & < & \ldots & < & b_k & < & b_1 + n \\ & & \vee & & \vee & & \ldots & & \vee & & \\ 1 & \leqslant & a_1 & < & a_2 & < & \ldots & < & a_k & \leqslant & n \end{array}$$

Proof idea for Ⓑ $\to$ Ⓒ: follow the right strands,
and encode permutation + winding info:

strand $a \to b$ goes around the cylinder w times $\rightsquigarrow a < b + w * n$.

Ⓑ $\tilde{n}$-webs on a cylinder;

Ⓒ n-close increasing couples of increasing integer sequences:

$$\begin{array}{ccccccccccc} & & b_1 & < & b_2 & < & \dots & < & b_k & < & b_1+n \\ & & \vee & & \vee & & \dots & & \vee & & \\ 1 & \leqslant & a_1 & < & a_2 & < & \dots & < & a_k & \leqslant & n \end{array}$$

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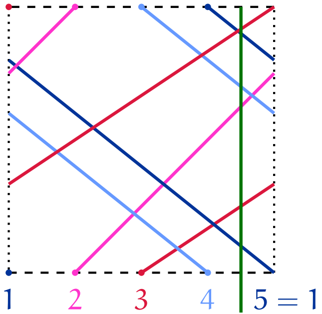
Example:



1	2	twists
2	1	
↑	↑	
2	3	

$$\rightsquigarrow \quad \begin{array}{ccc} 6 & < & 9 \\ \vee & & \vee \\ 2 & < & 3 \end{array}$$

II-sequences

Theorem (*L. ’21*): There are explicit bijections between:

Ⓑ $\tilde{n}$-webs on a cylinder;

Ⓒ n-close increasing couples of increasing integer sequences:

$$\begin{array}{ccccccccccc} & & b_1 & < & b_2 & < & \dots & < & b_k & < & b_1+n \\ & & \vee & & \vee & & \dots & & \vee & & \\ 1 & \leqslant & a_1 & < & a_2 & < & \dots & < & a_k & \leqslant & n \end{array}$$

Proof idea for Ⓑ → Ⓒ: follow the right strands,
and encode permutation + winding info:

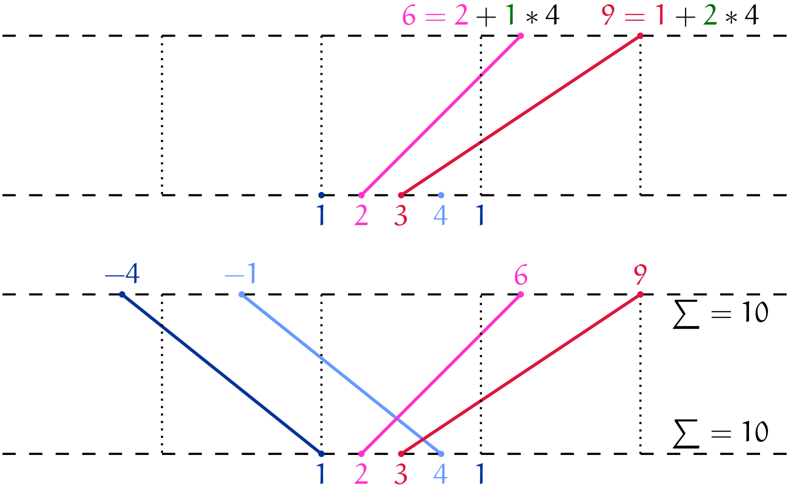
strand $a \to b$ goes around the cylinder w times $\rightsquigarrow a < b + w * n$.

Proof idea for Ⓒ → Ⓑ:

1. decode the permutation + winding info: Euclidean division;
2. draw the right strands (on the universal cover of the cylinder);
3. complete by the left strands and the vertical strands,
   use: the right winding nb = the left winding nb.

Example:

$$\begin{array}{ccc} 6 & < & 9 \\ \vee & & \vee \\ 2 & < & 3 \end{array}$$



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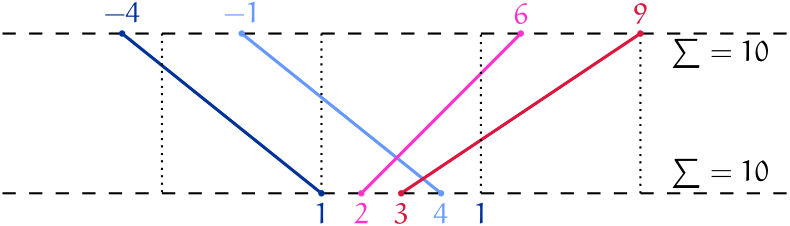

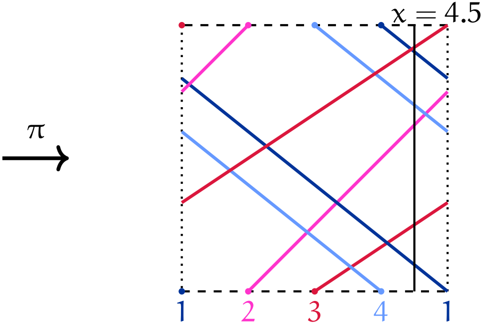

in C_4:

$x_4 x_3 x_1 x_4 x_2 x_1 x_3 x_2 x_4 x_3$

Answers for C_n

Theorem (*L. '21*): There are explicit bijections between:

Ⓐ the elements of C_n;

Ⓑ $\tilde{n}$-webs;

Ⓒ n-close increasing couples of increasing integer sequences.

Proof idea for Ⓐ → Ⓒ: use the C_n-chain $\sigma_i(a,b) = (a,a)$ for $i < n$, and $\sigma_n(a,b) = (a, a+n)$.

Example: $x_4x_3x_1x_4x_2x_1x_3x_2x_4x_3 \in C_4$.

$$(1,2,3,4) \overset{x_3}{\mapsto} (1,2,3,3) \overset{x_4}{\mapsto} (7,2,3,3) \overset{x_2}{\mapsto} (7,2,2,3) \overset{x_3}{\mapsto} (7,2,2,2) \overset{x_1}{\mapsto} (7,7,2,2)$$
$$\overset{x_2}{\mapsto} (7,7,7,2) \overset{x_4}{\mapsto} (6,7,7,2) \overset{x_1}{\mapsto} (6,6,7,2) \overset{x_3}{\mapsto} (6,6,7,7) \overset{x_4}{\mapsto} (11,6,7,7)$$

Modulo 4: $(3,2,3,3)$; right strands: $2 \to 2$ and $3 \to 1$.

Twists: $(1,2,3,4) \mapsto (11,6,7,7)$; $6 = 2 + 1 * 4$, $11 = 3 + 2 * 4$.

Sequences: $6 = 2 + 1 * 4$, $9 = 1 + 2 * 4$.

$$\begin{matrix} 6 & < & 9 \\ \vee & & \vee \\ 2 & < & 3 \end{matrix}$$

14 Answers for $\mathrm{C}_{\mathfrak{n}}$

Theorem (*L. '21*): There are explicit bijections between:

Ⓐ the elements of $\mathrm{C}_{\mathfrak{n}}$;

Ⓑ $\tilde{\mathfrak{n}}$-webs;

Ⓒ $\mathfrak{n}$-close increasing couples of increasing integer sequences.

Corollaries:

2 word problem: a linear solution Ⓐ $\to$ Ⓒ;

3 a quadratic normal form: Ⓐ $\to$ Ⓒ $\to$ Ⓑ $\to$ Ⓐ

or Ⓐ $\to$ Ⓒ $\xrightarrow[\text{process}]{\text{inductive}}$ Ⓐ.

15 Generalisations?

Problems:

- no diagrammatic interpretation for general graphs;
- for a generically oriented chain, different webs may represent equivalent words.

Example: $\underset{1}{\bullet} \longleftarrow \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet}$

relations: $x_1^2 = x_1,\ x_2^2 = x_2,\ x_3^2 = x_3, \qquad x_1x_3 = x_3x_1,$

$x_1x_2x_1 = x_2x_1x_2 = x_2x_1,\ x_2x_3x_2 = x_3x_2x_3 = x_2x_3$

$$x_2x_1x_3 \quad \sim \quad x_2x_1x_3x_2$$

$$x_2x_1x_3x_2 = x_2x_1x_2x_3x_2 = x_2x_1x_2x_3 = x_2x_1x_3$$