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Braces in Bracelet Bay

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Reflection equation
as a tool for studying
YBE solutions

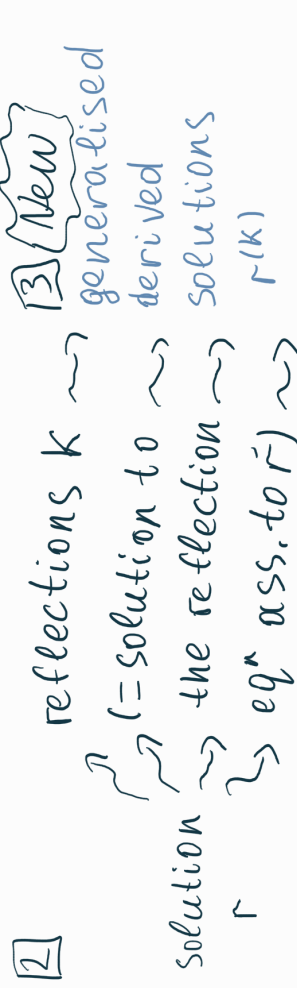
Hw Leandro Vendramin

① Short version

① solution $r \rightsquigarrow$ derived solution r^1



Secret goal: advertise diagrammatic calculus.



unique-like

Q2: Groups?

Ex.: (1) $r(Id) = r^1$ derived
(2) $r(*) = r$

Q1: Appropriate structure?
(A kind of semi-trusses?)

④ Reflections for involutive solutions
if time permits
& strange operations
 $(a \cdot b) \cdot a = b \cdot a$

② Long version



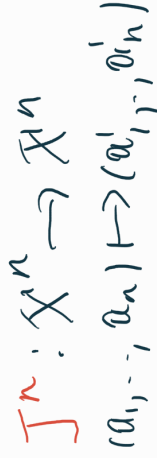
• YBE solution



• RND:

all ρ_b bijective

Guitar map:



$$a_1 a_2 \dots a_n$$

Thm: (1) J is 1:1

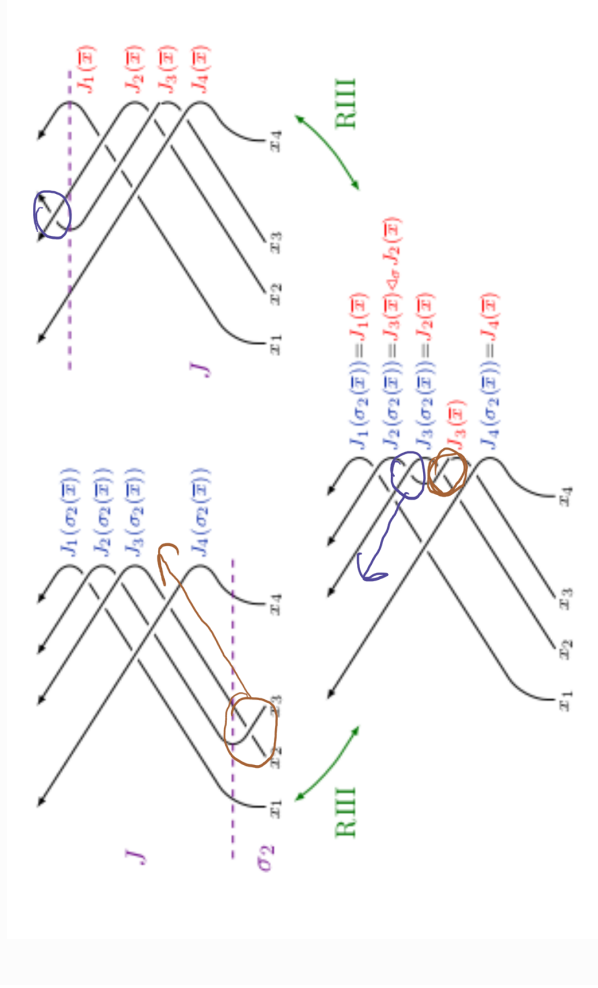
$$(2) J^{-1} = r^{-1} J$$

, where

$$\begin{matrix} & c \\ & \nearrow \\ a & \\ & \searrow \\ & b \end{matrix}$$

$$r^{-1}(a, b) = (c, a)$$

Proof:



Cre: (1) r^{-1} is a solution

(2) r is invertible / idempotent

r^{-1} is

(3) r and r^{-1} induce isomorphic actions $B_n^{(H)} \curvearrowright X^n$

(4) J induces a bijection

$$M(X, r) \xrightarrow{1:1} M(X, r^{-1})$$

$\nearrow G(X, r) \xrightarrow{1:1} G(X, r^{-1})$ if r invertible and non-degenerate

$$M(X, r) = T(X) / r = \text{Id}$$

[2] Definition: r : solution

$K: X \rightarrow X$ is a **reflection** for r if

$$r K_2 r K_2 = K_2 r K_2 r : X \times X \rightarrow X \times X \quad (RE)$$

Physics:

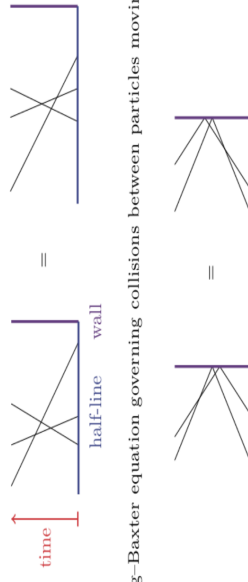
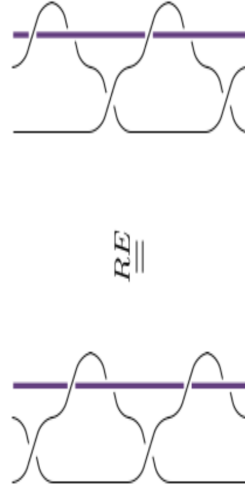


Fig. 1. The Yang-Baxter equation governing collisions between particles moving on a half-line.

Fig. 2. The reflection equation governing collisions between particles and the wall.

Topology:



[3]

k -Guitar map

$$J^{n,(k)}: X^n \rightarrow X^n$$

$$(a_1, \dots, a_n) \mapsto (a'_1, r, a'_n)$$



Thm: (1) J is 1:1
 (2) $J^{(k)} r = r^{(k)} J^{(k)}$, where



$$r^{(k)}(a, b) = (b', a')$$

Proof: analogous.

Cre: (1) $r^{(k)}$ is a solution

(2) r is invertible in volutive / idempotent

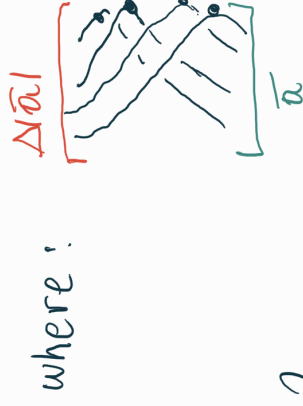
$$r^{(k)} \circ r = r$$

(3) r and $r^{(k)}$ induce isomorphic actions $B_n^+ \curvearrowright X^n$

(4) J induces a bijection

$$M(X, r) \xrightarrow{1:1} M(X, r^{(k)})$$

$$\text{Prop.: } J^{(k)}(\bar{a} \cdot \bar{b}) = J^{(k)}(p_{A/B}(\bar{a})) J^{(k)}(\bar{b})$$



where:

Iterate?

$$(r^{(1)})' = r'$$

$$(r^{(k)})^{(k')} = r^{(k)}$$

yields new solutions in general

$$\text{Thm.: } (r^{(k)})' = r'$$



Prop.: $b \cdot a = a_k$ extends to $M(X, r^{(k)}) \curvearrowright X$.

$$\text{Ex.: } \triangleleft = \triangleleft \text{ (structure shelf)}$$

