Braids & homology of algebraic structures: a round trip

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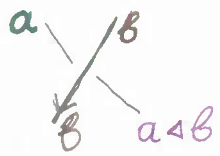
Plan

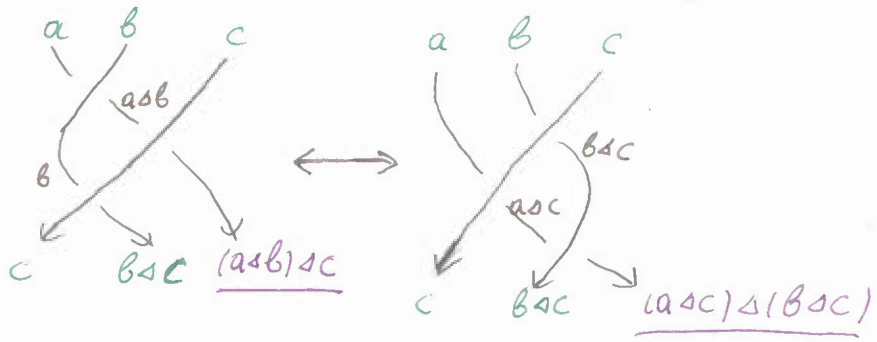
$$\begin{tikzcd}
\text{braided homology} \arrow[d, "③"] & & \\
\text{quandle homology} \arrow[rr, "①"] & & \text{knot \& braid theory} \arrow[llu, "②"']
\end{tikzcd}$$

① Quandle homology



diag. colorings by $(S, \triangleleft)$

R III



topology: pos. braids, braids, or. knots, unor. knots

- $\boxed{\text{RIII} \Leftrightarrow (a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)}$ (SD)
- RII $\Leftrightarrow \exists\, \widetilde{\triangleleft}$ s.t. $(a \triangleleft b) \widetilde{\triangleleft} b = a = (a \widetilde{\triangleleft} b) \triangleleft b$ (R)
- RI $\Leftrightarrow$ $a \triangleleft a = a$ (Q)
- orientat^n independence $\Leftrightarrow$ $(a \triangleleft b) \triangleleft b = a$ (K)

algebra: shelf, rack, quandle, kei

Ex.: (group G, $a \triangleleft b = b^{-1} a b$) is a quandle.

Theory: Joyce, Matveev:

$$\text{Knots}/_{K = -K^*} \hookrightarrow \text{Quandles}$$
$$K \longmapsto FQ(K)$$

Practice: "{Q-colorings of a knot diag. D} $= \mathrm{Hom}_{Qu}(FQ(K(D)), Q)$ is a knot invariant"

a "nice" quandle

$\#$ ↓

a number

↓

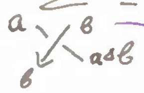
counting quandle invariants

Ex.: n-Fox colorings

↓ Boltzmann weights

$\{BW(Q\text{-col. of } D) \in \mathbb{R}\}$ is a knot invariand iff:

$BW = \sum \pm w(a \otimes b)$



QC I) $w(a \otimes a) = 0$

QC III) $w((\cancel{b \otimes c} - (a \triangleleft b) \otimes c + (a \triangleleft c) \otimes (b \triangleleft c)$
$- (\cancel{b \otimes c} - a \otimes c + a \otimes b)) = 0$

↓

quandle cocycle invariants

Rmk: can distinguish K from $-K^*$.

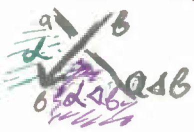
2 maps $V^{\otimes 3} \to V^{\otimes 2}$, $V := \mathbb{R}Q$

Gen^{ns}: (1) K^n in S^{n+2}

↓

2 maps $V^{\otimes n+2} \to V^{\otimes n+1}$

(2) $\sum \pm w(\alpha \otimes a \otimes b)$



↓

2 maps $M \otimes V^{\otimes 3} \to M \otimes V^{\otimes 2}$

② Braided homology

Ⓐ Motivation: parallel theories

SD structures	associative structures
$(Q,\triangleleft)$: quandle, $V=\mathbb{R}Q$	$(V,\cdot,1)$: unital associative algebra
$(a\triangleleft b)\triangleleft c=(a\triangleleft c)\triangleleft(b\triangleleft c)$ (SD)	$(v\cdot w)\cdot u=v\cdot(w\cdot u)$ (Ass)
	$1\cdot u=u\cdot 1=u$
M: Q-module	M: V-module
$(m\triangleleft a)\triangleleft b=(m\triangleleft b)\triangleleft(a\triangleleft b)$	$(m\cdot v)\cdot w=m\cdot(v\cdot w)$

presimplicial structure

$$\begin{cases} d_i : M\otimes V^{\otimes n}\to M\otimes V^{\otimes n-1}, \; 1\leq i\leq n \\ \text{s.t. } d_i\circ d_j=d_{j-1}\circ d_i \quad \forall i<j \end{cases} \Longrightarrow \quad \begin{array}{l} \partial_{n-1}\circ\partial_n=0 \\ \partial_n=\sum_{i=1}^{n}(-1)^{i-1}d_i \end{array}$$

$(m,a_1,\dots,a_n)\overset{d_i}{\longmapsto}(m\triangleleft a_i,a_1\triangleleft a_i,\dots,a_{i-1}\triangleleft a_i,a_{i+1},\dots,a_n)$

$m\otimes v_1\otimes\dots\otimes v_n\overset{d_i}{\longmapsto} m\otimes v_1\otimes\dots\otimes v_{i-1}\cdot v_i\otimes\dots\otimes v_n$

weakly simplicial structure

$$\begin{cases} s_i : M\otimes V^{\otimes n}\to M\otimes V^{\otimes n+1}, \; 1\leq i\leq n \\ \text{s.t. } d_j\circ s_i=\begin{cases} s_i\circ d_{j-1}, & j>i+1 \\ d_{i+1}\circ s_i, & j=i \\ s_{i-1}\circ d_j, & j<i \end{cases} \end{cases} \Longrightarrow \quad \left(\sum_{i=1}^{n}\mathrm{Im}(s_i),\partial_n\right) \text{ is a subcomplex of } (M\otimes V^{\otimes n},\partial_n)$$

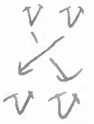
$\dots\overset{s_i}{\longmapsto}(m,a_1,\dots,a_{i-1},a_i,a_i,\dots,a_n)$

$\dots\overset{s_i}{\longmapsto} m\otimes v_1\otimes\dots\otimes 1\otimes v_i\otimes\dots\otimes v_n$

and many other properties...

(B) When parallel theories meet

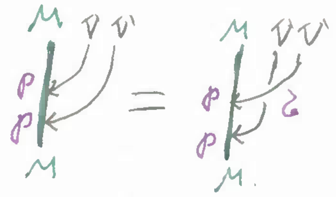
Braided vector space: $(V, \sigma: V\otimes V \to V\otimes V)$ s.t. $\underline{\sigma_1\circ\sigma_2\circ\sigma_1 = \sigma_2\circ\sigma_1\circ\sigma_2 : V^{\otimes 3}\circlearrowleft}$



(YBE)

Rmk: no invertibility conditions.

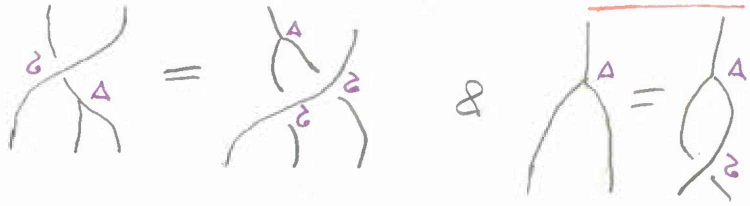
Braided V-module: $(M, \rho: M\otimes V \to M)$ s.t.



Left-braided coalgebra:

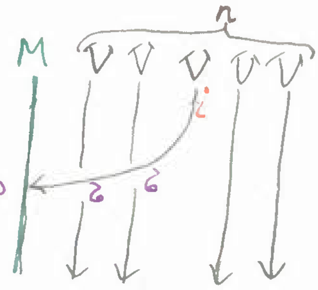
$(\underbrace{V, \sigma}_{\text{br. v. sp.}}, \Delta: V \to V\otimes V)$ s.t.

(cocomm)



Rmk: 3-valent knotted graphs & handle-body knots.

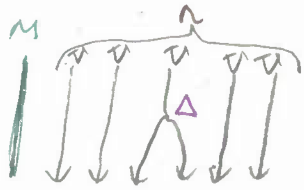
Th. (L., 2012): ① $d_i =$



$= \rho\circ\sigma_1\circ\ldots\circ\sigma_{i-1}$

defines a presimplicial structure on $M\otimes T(V)$.

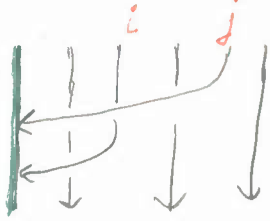
② $s_i =$



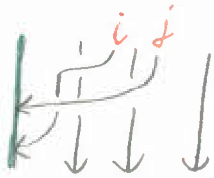
$= \Delta_i$

completes it into a weakly simplicial structure.

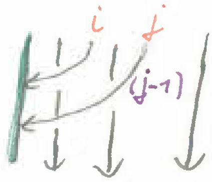
Proof:



$\underset{\text{(YBE)}}{=}$



$\underset{\text{Braided mod.}}{\overset{\text{def. of}}{=}}$



◇

Rmks: → right & 2-sided versions
→ functoriality
→ works in preadditive monoidal categories
→ quantum shuffles
→ sign in $\sum_i (-1)^{i-1} d_i$ is the intersection nb of the diagram.

Related constructions:

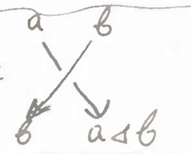
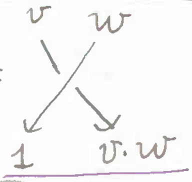
→ Majid: braided differential calculus $(\partial \circ \partial \neq 0)$
ex.: $V = \mathbb{R}x$, $\sigma(x \otimes x) = q\, x \otimes x$, $q \in \mathbb{R}^*$ $\rightsquigarrow \partial(x^{\otimes n}) = \underbrace{\frac{q^n - 1}{q-1}}_{\text{"}(n)_q} x^{\otimes n-1}$

→ Carter-Elhamdadi-Saito:
homology of the set-theoretic solutions to (YBE)

→ Eisermann:
Yang-Baxter cochain complex $\mathrm{Hom}_{\mathbb{R}}(V^{\otimes n}, V^{\otimes n})$

③ SD/ass. structure $\longmapsto$ braided v. sp.

classical ↘ homology theory ↙ Th.

SD structures	ass. algebras	Leibniz algebras
$\sigma_{SD} =$ 	$\sigma_{Ass} =$ 	$\sigma_{Lei}: v \otimes w \longmapsto w \otimes v + 1 \otimes [v,w]$
(YBE) $\Leftrightarrow$ (RIII) $\Leftrightarrow$ (SD) for $\triangleleft$	(YBE) $\Leftrightarrow$ (Ass) for $\cdot$ if $v \cdot 1 = v \ \forall v \in V$	(YBE) $\Leftrightarrow$ (Lei) for $[,]$ if $[v,1] = [1,v] = 0 \ \forall v \in V$ $[v,[w,u]] + [[v,u],w] =$ $= [[v,w],u]$
$\exists \sigma^{-1} \Leftrightarrow$ (R)	$\exists \sigma^{-1}$	$\exists \sigma$
br. module $\Leftrightarrow$ quandle module	br. module $\Leftrightarrow$ algebra module	br. module $\Leftrightarrow$ Leibniz module
$\Delta_{SD}: a \mapsto a \otimes a$ (Cocomm) $\Leftrightarrow$ (Q)	$\Delta_{Ass}: v \mapsto 1 \otimes v$ (Cocomm) $\Leftrightarrow$ $1 \cdot v = v$	$\Delta_{Lei}: v \mapsto 1 \otimes v + v \otimes 1, \ v \in V'$ $1 \mapsto 1 \otimes 1$ if V is split: $V = V' \oplus \mathbb{R}\cdot 1$, & $[V', V'] \subseteq V'$
braided homologies $\cup$ → 1-term distributive (Przytycki-Sikora) → rack (Fenn-Rourke-Sanderson) → quandle (Carter-Jelsovsky-Kamada-Langford-Saito)	br. hom. $\cup$ → bar → Hochschild	br. hom. $\cup$ → Leibniz (Loday, Cuvier) Leibniz ↓ Lie; $T(V)$ ↓ $\Lambda(V)$; Leibniz hom. ↓ Chevalley-Eilenberg hom. $[v,w] = -[w,v]$