Cabling for involutive YBE solutions, with applications to decomposability

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① Setting

- solution = finite ND involutive set-theoretic YBE solution (X, r)
- $n = \# X > 1$
- $r(x,y) = (\sigma_x(y), \tau_y(x))$
- $\mathcal{G}(X,r) :=$ the subgroup of S_X gen. by the σ_x, $x \in X$ (the permutation group of (X,r)); $= \mathcal{G}$
- indecomposable sol^n: $\mathcal{G} \curvearrowright X$ transitively
  $\Uparrow$
- primitive sol^n: no no-trivial decompos^n into blocks preserved by the $\mathcal{G}$-action

Qu.: (1) detect (2) construct (3) classify (4) glue

② Known results

(P) Cedó-Jespers-Okniński '22 (Thm) $\leftarrow$ Ballester-Bolinches '19 (Qu.):

$$\boxed{\{\text{primitive sol.}\}/_{iso} = \{\mathbb{Z}_p,\ r(a,b) = (b-1, a+1)\}} \quad \leftarrow p \text{ is prime}$$

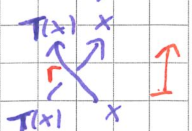
(1) ESS'99: For $n = p$, $\{\text{indec. sol.}\}/_{iso}$ $=$

(I) (3) Surprising fact: the map T knows smth about decomposability

First connection: a T-cycle $\subseteq$ a $\mathcal{G}$-orbit.

Diagonal map $T \in S_X : x \mapsto \tau_x^{-1}(x) \iff r(T(x), x) = (T(x), x)$

$\quad \rightsquigarrow$ cycles



T-partition of n, $\mathcal{P} = \mathcal{P}(X,r)$

Ex.: $\bullet\ \mathcal{P}(X, \text{flip}) = (1, 1, .., 1)$

$\bullet\ \mathcal{P}(\mathbb{Z}_n, r(a,b) = (b-1, a+1)) = (n)$.

$\uparrow$

$r(b-1, b) = (b-1, b)$, $\quad T(b) = b-1$

$\ast \rightarrow\ \bullet$ DS'23: $\Rightarrow \mathcal{P} = (pq), (p,..,p)$ or $(q,..,q)$ for indec. sol. of size $n = pq$

(2) Dietzel Sastriques '23, based on Cedó-Okniński '22: $\hookrightarrow$ square-free n

$\boxed{\text{complete classification of indec. sol. for } n = pq,\ p<q \text{ primes}}$

(4) Rump '05 (Thm) $\leftarrow$ Gateva-Ivanova '96:

$$\boxed{\mathcal{P} = (1, .., 1) \Rightarrow (X, r) \text{ decomposable}}$$

$\uparrow$

$r(x,x) = (x,x)$, square-free sol^n

Ramirez-Vendramin '21:

$\boxed{\rho = (n) \Rightarrow (X,r) \text{ indecomposable}}$

(5) Camp-Mora & Sastriques '21, R-V '21 } IMRN '21

- general
- advanced group theory
- uses Rump '05

part. cases

$\boxed{\gcd(\mathrm{order}(T), n) = 1 \Rightarrow (X,r) \text{ decomposable}}$

(6) R-V '21:

	n	T-partitions for indecomp. sol.					
•	4	(4)	(2,1,1)				
	6	(6)	(3,3)	(2,2,2)			
•	8	(8)	(4,4)	(2,2,2,2)	$\boxed{(6,2)}$	$\boxed{(4,2,2)}$	(2,2,1,1)
•	9	(9)	(3,3,3)	(3,3,1,1,1)			
	10	(10)	(5,5)	(2,2,2,2,2)			

(3) Our results (L.-R.-V. '22)

a Master student → $\boxed{\text{elementary machinery}}$ → decomposability criteria (DC)

Examples of DC:

(A) CMS Thm: elementary proof

DS'23 ⇒ (B) $n = pq$, $p \neq q$ prime, T has an s-cycle, $(p-1)q < s < pq$, $(s,p)=1$

Ex.: $n = 14 = 2\cdot 7$ $7 < s < 14$, $(s,2)=1$: $s = 9, 11, 13$

Ex: $\rho = (9,2,2,1) \Rightarrow (X,r)$ decomposable (CMS does not apply!)

$n = 15 = 3\cdot 5$, $10 < s < 15$, $(s,3)=1$: $s = 11, 13, 14$

(C) $n = ab$, $\rho = (a, c, c')$, a, b, c, c' pairwise coprime (except for c & c'), $c \geq b - a$

Ex.: $n = 2b$, $\rho = (2, b-4, b+2)$, $b \geq 5$ odd

(D) $n=2d$, d odd, $\rho=(2a,b,c)$, $(2d,abc)=1$, $b\leq c$, $c\neq 2a+b$

Ex.: $n=18$, $\rho=(10,7,1)$

DS'23 $\Rightarrow n=22$, $\rho=(10,9,3)$ or $(10,7,5)$

DS'23 $\Rightarrow$ (E) $n=pq$, $p<q$ prime, $\exists k\in\rho$ s.t. $q\nmid k$
or $\forall k\in\rho$, $p\nmid \ell$

Ex.: $n=2q$, q odd prime, (X,r) indec. $\Rightarrow \rho=(q,q)$ or $\exists$ even term

Qu.: Find all possible T-partitions of indec. solutions?

④ Cabling

$(X,r) \rightsquigarrow (G=G(X,r),\circ,+)$ structure brace $\rightsquigarrow$ infinite solⁿ (G,r_G)

- $(G,\circ)=\langle X \mid x\circ y=\sigma_x(y)\circ\tau_y(x),\ x,y\in X\rangle$ structure group
- $(G,+)=$ free ab. group on X.

$\iota_k : X \hookrightarrow G(X,r)$

$x \longmapsto k\cdot x := \underbrace{x+\dots+x}_{k \text{ times}}$

Thm: $\iota_k(X)$ is a subsolⁿ of (G,r_G)

□ $r_G(kx,\ell y)=(\ell\,\sigma_{kx}(y),\ k\,T^{k-1}\tau_{\ell y}T^{-k+1}(x))$ ☒

$$\begin{array}{ccc} X & \xhookrightarrow{\iota_k} & G \\ \underbrace{r^{(k)}}_{} & \longleftarrow & r_G \end{array}$$

k-cabling of (X,r)

$r^{(k)}(x,y)=(\sigma_{kx}(y),\ T^{k-1}\tau_{ky}T^{-k+1}(x))$

- $r^{(1)}=r$
- for a suitable d, $r^{(d)}=$ flip & $r^{(k+d)}=r^{(k)}$ $\forall k$

Properties:
- the diagonal map of $r^{(k)}$ is T^k
- $\mathcal{G}(X,r^{(k)})$ is a subgroup of $\mathcal{G}$

Thm: $\rightarrow$
- $\begin{cases} x\in \mathcal{G}\text{-orbit of size } m \\ x\in \mathcal{G}(X,r^{(k)})\text{-orbit of size } m' \end{cases}$ $\Rightarrow m'$ is a multiple of the max. divisor m_k of m coprime with k

Ex.: $k=\mathrm{ord}(T) \Rightarrow T^{(k)}=\mathrm{Id} \overset{\text{Rump'05}}{\Rightarrow} (X,r^{(k)})$ decomposable
If $(k,n)=1$, then $m\mid n \Rightarrow (k,m)=1 \Rightarrow m_k=m \Rightarrow m'=m$.
$\Rightarrow \mathcal{G}$-orbits $=\mathcal{G}(X,r^{(k)})$ orbits $\Rightarrow (X,r)$ decomposable (CMS'21).

- $(r^{(k)})^{(k')} = r^{(kk')}$
- $\mathrm{Ret}(X', r^{(k)})$ = a quotient of $\mathrm{Ret}(X', r)^{(k)}$

Rmk.

⑤ Other applications of cabling

$\boxed{A}$ Structure of braces

brace $B \rightsquigarrow B^{(k)} := \{ka \mid a \in B\}$ (sub-brace) $\underset{\text{left ideal}}{:}$ (qu. of Lorenzo Stefanello)
- $\sigma_x(ky) = k\sigma_x(y)$
- $ka \circ kb = k((ka)\circ b + (k-1)a)$

$\rightsquigarrow$ 1) A quick proof of the solvability of $(B, \circ)$ for a finite brace (ESS'99):

$\#B = ab,\ (a,b)=1 \rightsquigarrow \#B^{(a)} = b \Rightarrow (B^{(a)}, \circ)$ a b-Hall subgroup of $(B, \circ)$

2) $(B,+) = \mathbb{Z}_n,\ d \mid n \Rightarrow \exists\, \underset{\text{subgroup}}{D \leq (B,\circ)},\ \#D = d \qquad \rightsquigarrow B^{\#B/d}$

$\boxed{B}$ Dehornoy class

$d := \mathrm{lcm}\left\{ \underset{\text{in } (\mathcal{G},+)}{\mathrm{ord}(\sigma_x)} \;\middle|\; x \in X' \right\}$ $\qquad$ Our result: this conceptual def. of d.

($\mathcal{G}$: finite ab. group)

Rmk: $\mathrm{ord}(\sigma_x) = \mathrm{ord}(\sigma_{g\cdot x}) \quad \forall g \in G$

Chouraqui–Godelle '14 (part. case $d=2$), Dehornoy '15:

$\boxed{G^{(d)} \leq Z(G,\circ),\ \text{free ab., of finite index } d^n}$

$G \twoheadrightarrow G/G^{(d)} \twoheadrightarrow \mathcal{G}$ $\quad$ braces $\qquad \Rightarrow \boxed{\mathrm{ord}(T) \mid d \mid \#\mathcal{G} \mid d^n}$

$\Rightarrow \mathrm{PrimeDiv}(\#\mathcal{G}) = \mathrm{PrimeDiv}(d)$

~~CMS: $\exists$ prime $p \nmid n$ & $p \mid \mathrm{ord}(T) \mid \#\mathcal{G}$~~

Qu. (Okniński): $\mathrm{PrimeDiv}(\#X') \overset{?}{=} \mathrm{PrimeDiv}(\#\mathcal{G})$? for (X, r) indec.

$\subseteq$: TRUE: $\mathcal{G} \curvearrowright X'$ transitively $\Rightarrow n = \#X = \#\{\text{orbit}\} \mid \#\mathcal{G}$.

$\supseteq$: FALSE: $\exists (X,r)$ indec., $n = \#X = 8$, $p = (6,2) \Rightarrow \mathrm{ord}(T) = 6 \Rightarrow 6 \mid \#\mathcal{G}$.

In fact, $\#\mathcal{G} = 24$, $(\mathcal{G}, \cdot) \cong S_4$ $\qquad d = 6$.

$(\mathcal{G}, +) \cong C_6 \times C_2^2$

Conj (Feingesicht '23): $\max\{d(X', r) \mid \#X' = n\} = \max\{\prod_{i=1}^{k} n_i \mid k \in \mathbb{N},\ 1 \leq n_1 < \ldots < n_k,\ \sum_{i=1}^{k} n_i = n\}$
True for $n \leq 10$.