The Yang-Baxter equation, braces and Thompson's group F

Algebra Days in Caen 2022: from Yang–Baxter to Garside, Caen 2022

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The quantum Yang-Baxter equation - QYBE

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The QYBE is the equality $R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$ of linear transformations on $V \otimes V \otimes V$, where R^{ij} means R acting on the $i-$th and $j-$th components.

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A set-theoretical solution (X, r) of the QYBE [Drinfeld]

- V is a vector space spanned by a set X.
- R is the linear operator induced by a mapping $r: X \times X \rightarrow X \times X$, that satisfies $r^{12}r^{23}r^{12} = r^{23}r^{12}r^{23}$.

Properties of a solution (X, r)

Let $X = \{x_1, ..., x_n\}$ and let r be defined in the following way: $r(i, j) = (\sigma_i(j), \gamma_j(i))$, where $\sigma_i, \gamma_i : X \to X$.

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- (X, r) is braided $\Leftrightarrow$ $\sigma_i\sigma_j = \sigma_{\sigma_i(j)}\sigma_{\gamma_j(i)}$ and $\gamma_j\gamma_i = \gamma_{\gamma_j(i)}\gamma_{\sigma_i(j)}$ and $\gamma_{\sigma_{\gamma_j(i)}(k)}\sigma_i(j) = \sigma_{\gamma_{\sigma_j(k)}(i)}\gamma_k(j)$, $1 \leq i,j,k \leq n$.

The QYBE group: the structure group of (X, S)

Assumption: The pair (X, r) is non-degenerate, involutive and braided. It is called a non-degenerate, involutive set-solution.

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There are exactly $\frac{n(n-1)}{2}$ *defining relations.*

Example

Let $X = \{x_0, x_1, x_2, x_3\}$.

$$\sigma_0 = (0)(1)(2,3) \qquad \sigma_1 = (1,2,0,3)$$
$$\sigma_2 = (2)(3)(0,1) \qquad \sigma 3 = (1,3,0,2) \tag{1}$$

The solution is indecomposable with defining relations:

$$x_1 x_1 = x_2 x_0 \quad x_1 x_0 = x_3 x_2$$
$$x_0 x_3 = x_2 x_1 \quad x_1 x_2 = x_0 x_1 \tag{2}$$
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The center of G is generated by $\Delta = (x_0 x_1)^2 = (x_2 x_3)^2$

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An inverse semigroup is a regular semigroup in which all the idempotents commute: the set $E(S)$ of idempotents of an inverse semigroup S is a commutative subsemigroup.
$E(S)$ is ordered by $e \leq f$ iff $ef = e = fe$.

Commutative inverse monoids

Some definitions (X a set)

- A *partial function* of X is a function f between two (non-necessarily proper) subsets of X, the domain and the range of f are denoted by $\mathcal{D}_f$, and $\mathcal{R}_f$ respectively.

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- The identity is 0_X, the zero function on X.

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- There is a zero element: the vacuous map $\emptyset \to \emptyset$.
- The idempotents of I_X are the partial identities on X.

Operations on inverse monoids

Let S be a semigroup, M be a monoid

M is said *to act on* S *(on the left) (by endomorphisms)* if there exists a map $M \times S \to S$, denoted by $(a, s) \mapsto a \bullet s$ satisfying the following conditions:

- for any $a, b \in M$, $s \in S$, $(ab) \bullet s = a \bullet (b \bullet s)$.
- for any $a \in M$, $s, s' \in S$, $a \bullet (ss') = (a \bullet s)(a \bullet s')$.
- for every $s \in S$, $1 \bullet s = s$.

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Let M act on S (on the left) (by endomorphisms)

The semidirect product $S \rtimes M$ is not an inverse semigroup!!!

The restricted product of inverse semigroups

Let M, S be inverse semigroups. Let $E(M)$ denote the set of idempotents of M (ordered by $e \leq f$ if and only if $ef = e = fe$). Assume the following assumptions:

- M acts on S by endomorphisms.
- There exists a surjective homomorphism $\epsilon : S \to E(M)$.
- For each $s \in S$, there exists $\epsilon(s) \in E(M)$ such that

$$\epsilon(s) \leq e \iff e \bullet s = s, \forall e \in E(M)$$

Let $S \bowtie M$ be the following set with the binary operation defined below:

$$S \bowtie M = \{(s, m) \in S \times M \mid r(m) = \epsilon(s)\}$$

$$(s, m)(s', m') = (\, s(m \bullet s'),\, mm')$$

$S \bowtie M$ is an inverse semigroup.

Definition and properties of a partial solution

Let $X \neq \emptyset$ be a set. Let $\mathcal{D}, \mathcal{R} \subseteq X \times X$.

Define $r : \mathcal{D} \to \mathcal{R}$, by $r(x, y) = (\sigma_x(y), \gamma_y(x))$, where $\sigma_x : \mathcal{D}_{\sigma_x} \to \mathcal{R}_{\sigma_x}$, $\gamma_y : \mathcal{D}_{\gamma_y} \to \mathcal{R}_{\gamma_y}$; $\mathcal{D}_{\sigma_x}, \mathcal{R}_{\sigma_x}, \mathcal{D}_{\gamma_y}, \mathcal{R}_{\gamma_y} \subseteq X$

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- (X,r) is *square-free*, if $\forall x \in X$, $(x,x) \in \mathcal{D}$ and $r(x,x) = (x,x)$.

An example of square-free partial solution

If (X, r) is braided, we call (X, r) *a partial set-theoretic solution*. If (X, r) is a non-degenerate, involutive partial set-theoretic solution, we call it *a partial solution*.

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(X, r) is a square-free partial solution, with:

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The structure inverse monoid of a partial solution

Let (X, r) be a partial set-theoretic solution.

- The *structure group* of (X, r) is $G(X, r) = \mathrm{Gp}\langle X \mid \; xy = \sigma_x(y)\gamma_y(x) \; ; \; (x, y) \in \mathcal{D}\rangle$.

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The structure group of a trivial partial solution is a partially commutative group (or a right-angled Artin group)

A partial solution (X, r) is *trivial* if for every $x \in X$, $\sigma_x = \mathrm{Id}_{\mathcal{D}_{\sigma_x}}$, $\gamma_x = \mathrm{Id}_{\mathcal{D}_{\gamma_x}}$.

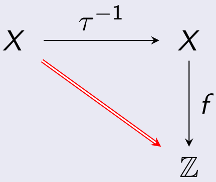
Properties of square-free partial solutions 1

Let (X, r) be a partial set-theoretic solution.

I_X: the symmetric inverse monoid.
A: the commutative inverse monoid (partial $f : X \to \mathbb{Z}$, finite support).

Let $\tau \in I_X$ and $f \in A$, $f : X \to \mathbb{Z}$ a partial function

I_X acts (totally) on A by endomorphisms: $\tau \bullet f = f \circ \tau^{-1}$



Properties of square-free partial solutions 2

Let $x \in X$, $g, h \in \mathrm{IM}(X, r)$, $f \in A$:

1. The following map is a homomorphism of monoids:

$$\alpha : \mathrm{IM}(X, r) \to \mathrm{I}_X \quad x \mapsto \sigma_x$$

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Characterization of square-free partial solutions

Theorem 1 [F.C]

$$\pi : \mathrm{IM}(X, r) \to A$$
$$x_i \mapsto \delta_i$$
$$\pi(gh) = \pi(g) + g \bullet \pi(h)$$

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Furthermore, $\delta_x(y)$ is not defined for $y \in X \setminus \mathcal{R}_{\sigma_x}$.

Characterization of square-free partial solutions

Theorem 2 [F.C] The restricted product $A \bowtie \mathrm{I}_X$ is defined by:

$$A \bowtie \mathrm{I}_X = \{(f,\tau) \in A \times \mathrm{I}_X \mid \mathcal{R}_\tau = \mathcal{D}_f\}$$
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Theorem 3 [F.C]

Let (X,r) be a square-free partial solution, with $\mathrm{IM}(X,r)$.

$$\psi : \mathrm{IM}(X,r) \to A \bowtie \mathrm{I}_X$$
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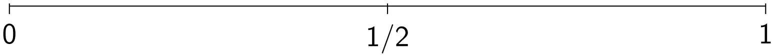
Furthermore, $\mathrm{Im}(\psi)$ is an inverse monoid.

Dyadic subdivisions



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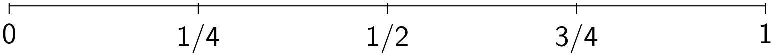

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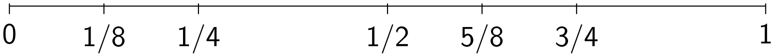

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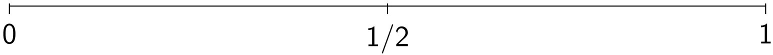

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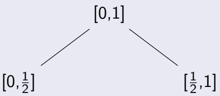
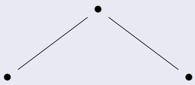
To each dyadic interval there corresponds a binary tree:





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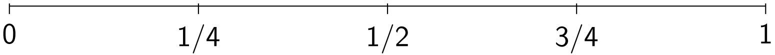

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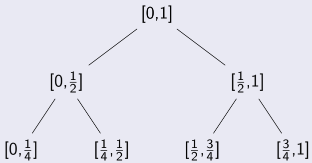

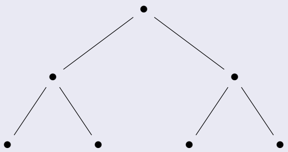

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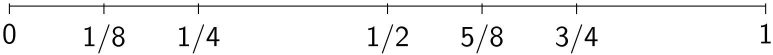

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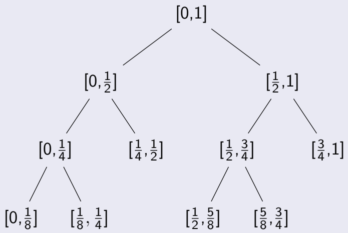

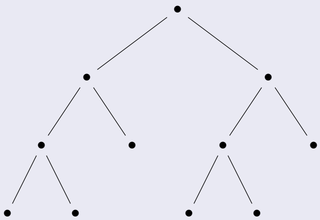

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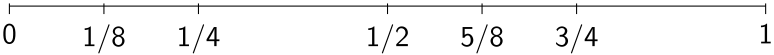

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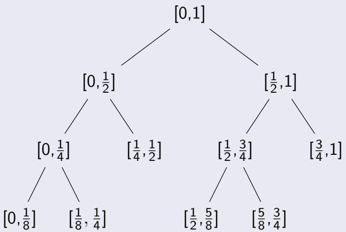

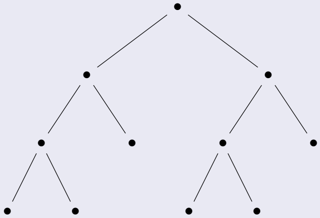

Introduction to Thomson group F (1)

Given $\mathcal{D}$, $\mathcal{R}$, with the same number of cuts, a *dyadic rearrangement of* $[0,1]$ is a picewise-linear $f : [0,1] \to [0,1]$ that sends each interval of $\mathcal{D}$ linearly onto the corresponding interval of $\mathcal{R}$. The set of all dyadic rearrangements forms a group under composition: the Thompson group F.

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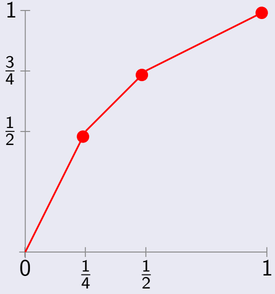
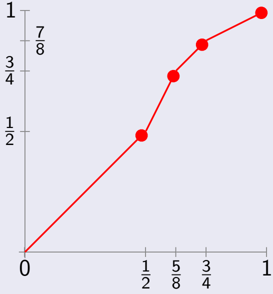
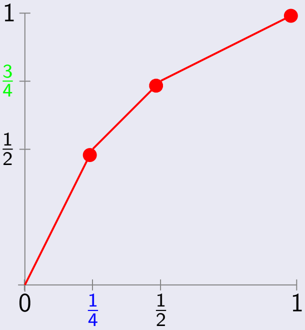
Dyadic rearrangements for x_0 at left and x_1 at right

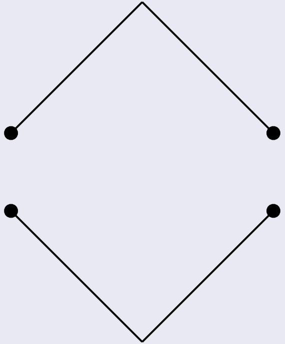




Introduction to Thomson group F (2)

The dyadic rearrangement and tree diagram for x_0





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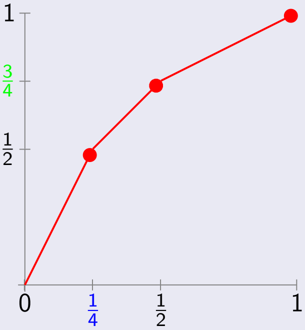
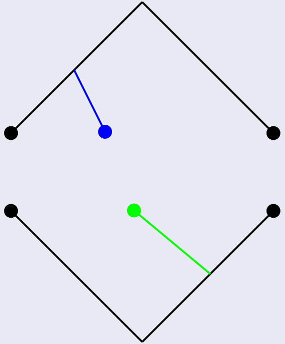
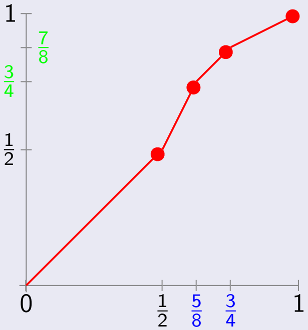
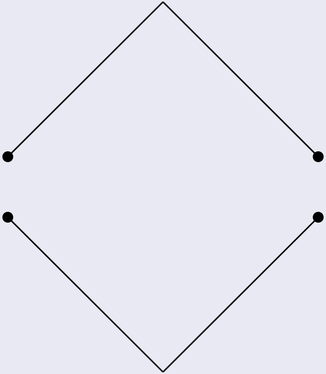




Introduction to Thomson group F 3

The dyadic rearrangement and tree diagram for x_1





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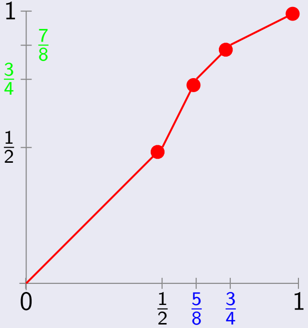

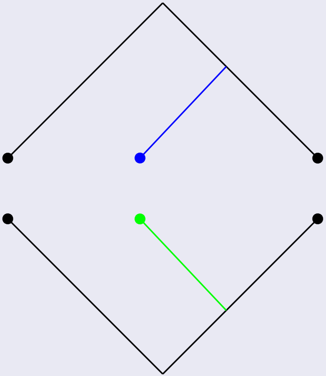

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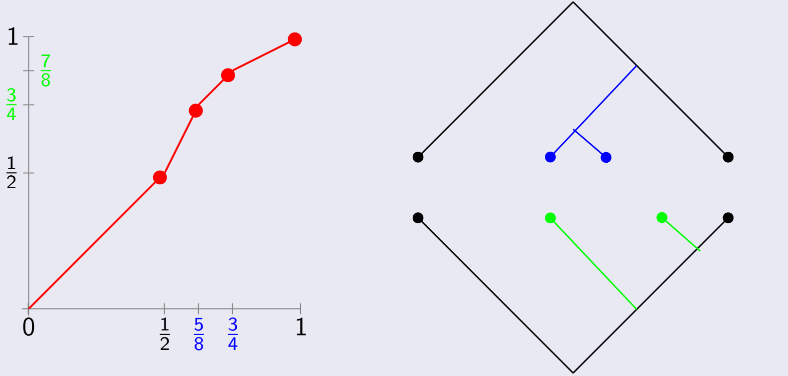

Several presentations of Thompson group F

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An infinite presentation of Thompson group F

$$\langle x_0, x_1, x_2, ... \mid x_nx_k = x_kx_{n+1},\ k < n \rangle$$

$$x_n = x_0x_{n-1}x_0^{-1} = x_0^{n-1}x_1x_0^{-(n-1)}$$

F as the structure group of a partial solution (1)

Definition of a partial solution

Let $X = \{x_0, x_1, x_2, ...\}$. Let $\sigma_n : X \to X$ and $\gamma_n : X \to X$ be the following partial functions.

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$$\mathcal{D}_{\sigma_n} = X \setminus \{x_{n+1}\} \qquad \mathcal{R}_{\sigma_n} = X$$
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F as the structure group of a partial solution $\mathcal{F}$ (2)

Definition of a partial solution $\mathcal{F}$

Let define the following partial function:

$$\begin{aligned} &r : X \times X \to X \times X \\ &r(x_i, x_j) = (x_{\sigma_i(j)}\, ,\, x_{\gamma_j(i)}) \end{aligned} \tag{3}$$

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Lemma

(X, r) is a square-free, non-degenerate, involutive partial set-theoretic solution, denoted by $\mathcal{F}$.

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Theorem [F.C]

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2. $\mathrm{IM}(X, r)$, the structure inverse monoid of $\mathcal{F}$, embeds into the inverse monoid $A \bowtie \mathrm{I}_X$, where A is the commutative inverse monoid $\{f : \mathcal{D}_f \to \mathbb{Z} \mid \mathcal{D}_f \subseteq X\}$, with pointwise operation, and I_X is the inverse symmetric monoid.

Some remarks to conclude

- $G(X,r)$, with X finite, is a Garside group. Garside groups are torsion-free and biautomatic. F is also torsion-free, but it is not known wether it is automatic.

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- The centre of the structure group of an indecomposable solution (X,r), with X finite, is cyclic. $Z(F)=\{1\}$.

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- The centre of the structure group of an indecomposable solution (X, r), with X finite, is cyclic. $Z(F) = \{1\}$.
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- What can be said about the other Thompson's groups F, T, V, with $F \subset T \subset V$?

The end

Thank you!