Crossed modules and beyond

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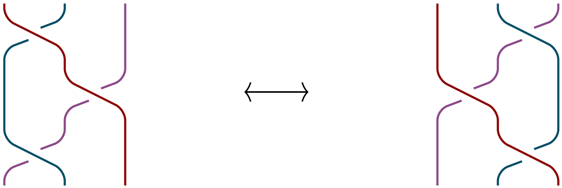
Joint work with **Friedrich WAGEMANN**



Leeds, 6 July 2016

1 Yang–Baxter equation

✓ Object V in a monoidal category (e.g. vector space / set).

✓ $\sigma \colon V^{\otimes 2} \to V^{\otimes 2}$.

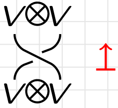
Yang–Baxter equation (YBE):

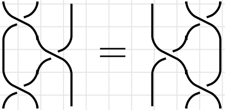
$$\sigma_1 \circ \sigma_2 \circ \sigma_1 = \sigma_2 \circ \sigma_1 \circ \sigma_2 \colon V^{\otimes 3} \to V^{\otimes 3}$$

where $\sigma_i = \mathrm{Id}_V^{\otimes i-1} \otimes \sigma \otimes \mathrm{Id}_V^{\otimes \cdots}$.

A map σ satisfying YBE is a braiding.

$\sigma \longleftrightarrow$





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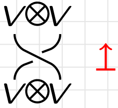

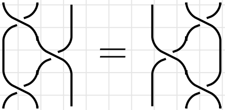

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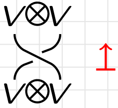
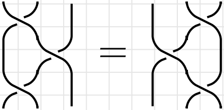
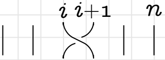
invertible braiding σ on V $\rightsquigarrow$

$\sigma \longleftrightarrow$ 



(Reidemeister III)

$B_n \to \mathrm{End}(V^{\otimes n})$



$\mapsto \sigma_i$


$\mapsto \sigma_i^{-1}$

2 Solutions to the YBE

1 Yetter–Drinfel′d module over a Hopf algebra H:

✓ vector space M;

✓ H-action $\rho\colon m \otimes h \mapsto m * h$;

✓ H-coaction $\delta\colon m \mapsto m_{(0)} \otimes m_{(1)}$;

✓ compatibility: actions and coactions can be switched

$$(m * h)_{(0)} \otimes (m * h)_{(1)} = m_{(0)} * h_{(2)} \otimes s(h_{(1)}) m_{(1)} h_{(3)}$$

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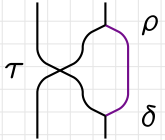

switching

$+$ a toll

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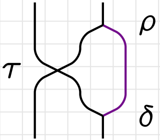

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✦ All invertible f.-d. braidings.

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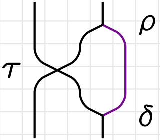

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✦ All invertible f.-d. braidings.

✦ $\mathcal{YD}_H^H$ has nice categorical features: braided monoidal, and even modular when $H = \Bbbk G$ for a finite group G

$\rightsquigarrow$ link and 3-mld invariants.

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2 Self-distributive set (= shelf):

✓ set S;

✓ self-distributive binary operation $\lhd$:

$$(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$$

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✦ $\mathbb{Z}[t]$-module with $a \lhd b = ta + (1-t)b$.

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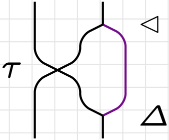

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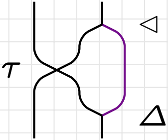

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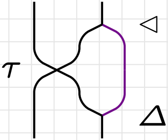

Rack axiom $\iff$ Reidemeister II move;

Quandle axiom $\iff$ Reidemeister I move.

$\rightsquigarrow$ Invariants of braids, links, knotted surfaces & graphs.

2 Solutions to the YBE

3 Crossed module of groups:

✓ group morphism $\pi\colon K \to G$;

✓ G-action $\cdot$ on K by group automorphisms;

✓ compatibility:

$$\pi(k \cdot g) = g^{-1}\pi(k)g, \qquad k \in K,\ g \in G,$$

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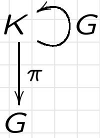

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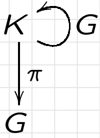

Representation of $(K, G, \pi, \cdot)$:

✓ $(M = \bigoplus_{k\in K} M_k) \curvearrowleft G$; ✓ $M_k * g = M_{k\cdot g}$.

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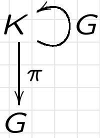

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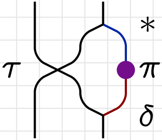
$$\sigma_{CrMod}(m \otimes n) = \sum_{k \in K} n_k \otimes m * \pi(k)$$



switching + a toll + currency exchange

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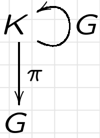

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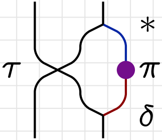


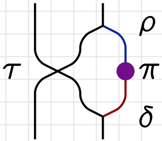
switching + a toll + currency exchange

✦ The category $\mathcal{M}(K, G, \pi, \cdot)$ is ✓ braided monoidal;

✓ pre-modular when G and K are finite;

✓ modular if moreover π is an isomorphism.

2 Solutions to the YBE



Question: A unified treatment of these braidings?

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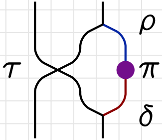

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Three levels of braidings.

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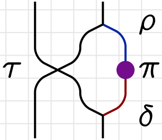

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Bonuses:

✦ new sources of braidings;

✦ categories with interesting associators.

3 Braided vocabulary

Rank r braided system (in $\mathcal{C}$):

✓ objects $V_1, V_2, \ldots, V_r$;

✓ (multi-)braiding $\sigma^{i,j}\colon V_i \otimes V_j \to V_j \otimes V_i$,

$$1 \leq i \leq j \leq r;$$

✓ compatibility: colored YBEs

$$\boxed{\sigma_1^{j,k} \circ \sigma_2^{i,k} \circ \sigma_1^{i,j} = \sigma_2^{i,j} \circ \sigma_1^{i,k} \circ \sigma_2^{j,k}}$$

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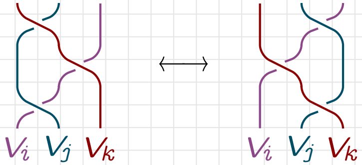

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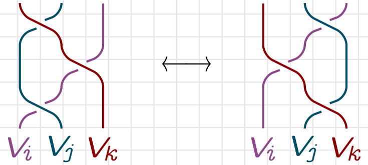

Braided object = rank 1 braided system.

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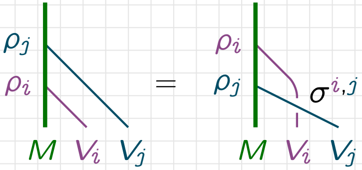

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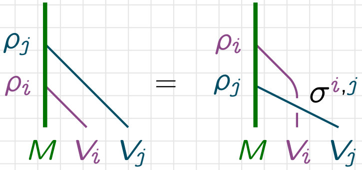

Categories $\mathbf{Mod}_{(\overline{V};\overline{\sigma})}$, $\mathbf{Mod}^{(\overline{V};\overline{\sigma})}$, etc.

4 Examples of braided systems

✓ Unital associative algebra (A, μ, ν) $\rightsquigarrow$

$$\boxed{\sigma_{Ass} = \nu \otimes \mu}$$

in $\mathbf{Vect}_{\mathbb{k}}$: $\sigma_{Ass}(v \otimes v') = 1 \otimes vv'$

- YBE for σ_{Ass} $\iff$ the associativity of μ.
- Usual A-modules are braided modules over $(A; \sigma_{Ass})$.

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✓ Shelf $(S, \lhd) \rightsquigarrow$

$$\boxed{\sigma_{SD}(a, b) = (b, a \lhd b)}$$

- ✦ YBE for $\sigma_{SD} \iff$ the self-distributivity of $\lhd$.
- ✦ $\exists \sigma_{SD}^{-1} \iff (S, \lhd)$ is a rack.

4 Examples of braided systems

✓ Unital Lie algebra $(L, [\,], 1)$, $[v, 1] = [1, v] = 0$.

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Basic alg. relations are instances of the YBE!

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✦ Usual L-modules are braided modules over $(L; \sigma_{Lie})$.

It is enough to take a unital Leibniz algebra
(= non-symmetric Lie).

4 Examples of braided systems

✓ F.-d. Hopf algebra H $\rightsquigarrow$ two rank 2 braided systems $(H, H^*; \overline{\sigma})$, and a rank 4 one.

- ✦ cYBEs $\Longleftrightarrow$ bialgebra axioms.
- ✦ $\exists (\sigma^{i,j})^{-1}$ $\Longleftrightarrow$ H admits an antipode.
- ✦ Hopf modules / YD modules / Hopf bimodules are braided modules.

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✓ Poisson algebra P $\rightsquigarrow$ a rank 2 braided system $(P, P; \overline{\sigma})$.

5 Generalized YD modules

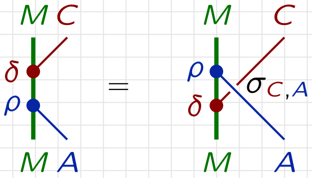
Yetter–Drinfel′d module over a braided system

$(C, A; \sigma_{A,A}, \sigma_{C,C}, \sigma_{C,A})$:

- ✓ object M;
- ✓ $(A; \sigma_{A,A})$-module structure ρ;
- ✓ $(C; \sigma_{C,C})$-comodule structure δ;
- ✓ compatibility: actions and coactions can be switched

$$\delta \circ \rho = (\rho \otimes \mathrm{Id}_C) \circ (\mathrm{Id}_M \otimes \sigma_{C,A}) \circ (\delta \otimes \mathrm{Id}_A)$$

in $\mathbf{Vect}_{\mathbb{k}}$: $(m * a)_{(0)} \otimes (m * a)_{(1)} = m_{(0)} * \tilde{a} \otimes \widetilde{m_{(1)}}$



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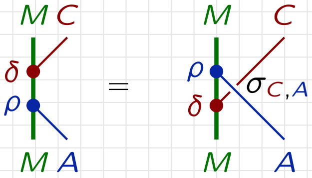

Category $\mathcal{YD}_A^C$.

✦ Relation to entwining structures, distributive laws, bimodules over a bimonad.

6 Generalized YD braidings

Theorem (L.-W. 2015): YD braidings generalize to $\mathcal{YD}_A^C$

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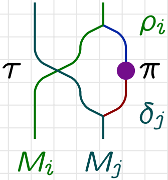
$$\rightsquigarrow \quad \text{Functors } (\mathcal{YD}_A^C)^{\times r} \to \mathbf{BrSyst}_r,$$

$$\big((M_i, \rho_i, \delta_i)\big) \mapsto (M_i; \sigma_{gYD}^{i,j}),$$

where $\boxed{\sigma_{gYD}^{i,j} = (\mathrm{Id}_{M_j} \otimes \rho_i) \circ (\tau \otimes \pi) \circ (\mathrm{Id}_{M_i} \otimes \delta_j)}$.



switching + a toll + currency exchange

Generalized YD modules: examples

1. GYD recover YD

Hopf algebra $(H, \mu, \nu, \varepsilon, \Delta, S)$ $\rightsquigarrow$ braided system

✓ $C = A = H$;

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$$\boxed{\mathcal{YD}_A^C \hookleftarrow \mathcal{YD}_H^H}$$

$$\sigma_{gYD} \leftarrow \sigma_{YD}$$

Generalized YD modules: examples

3 GYD recover reps of crossed modules

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GYD pave a way to new braidings

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6 Crossed module of shelves:

✓ shelf morphism $\pi\colon R \to S$;

✓ shelf action $\cdot$ of S on R by shelf morphisms;

✓ compatibility:

$$\pi(r \cdot s) = \pi(r) \lhd s, \qquad r \in R,\ s \in S,$$
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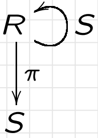

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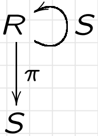

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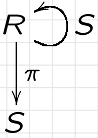

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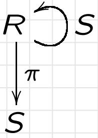

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✓ components: $C = R$, $A = S$;

✓ braiding:

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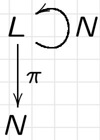
9 Crossed modules of Lie algebras

5 Crossed module of Lie (Leibniz) algebras:



✓ Lie algebra morphism $\pi \colon L \to N$;

✓ Lie action $[\,]$ of N on L by derivations:

$$[l,[n,n']] = [[l,n],n'] - [[l,n'],n],$$

$$[[l,l'],n] = [l,[l',n]] + [[l,n],l'];$$

✓ compatibility:

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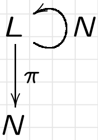

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A representation of $(L,N,\pi,[\,])$:

✓ $M \curvearrowleft N$;

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Example: Lie algebra $(L,[\,]) \rightsquigarrow$ cr. mod. $(L,L,\mathrm{Id}_L,[\,])$,

with a representation $L^+ = L \oplus \Bbbk 1$,

$$l * l' = [l,l'], \qquad 1 * l = 0,$$

$$\delta_0(l) = 1 * l, \qquad \delta_0(1) = 0.$$

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✦ The braiding $\sigma_{CrModLA}$ generalizes σ_{Lie}.

10 Categorical center

Braided system $(\mathcal{C}, A; \overline{\sigma}) \rightsquigarrow \mathcal{Z}_A^{\mathcal{C}}$ = the category of braided systems $(\mathcal{C},\ M,\ A;\ \overline{\sigma},\ \sigma_{\mathcal{C},M},\ \mathrm{Id}_{M\otimes M},\ \sigma_{M,A})$.

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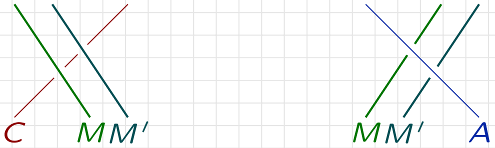
Theorem (L.-W. 2015): GYD modules are many

✦ $\mathcal{Z}_A^C$ is strict monoidal:

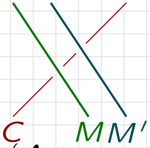


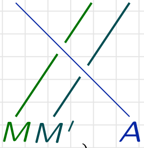
10 Categorical center

Braided system $(C, A; \overline{\sigma}) \rightsquigarrow \mathcal{Z}_A^C$ = the category of braided systems $(C,\ M,\ A;\ \overline{\sigma},\ \sigma_{C,M},\ \mathrm{Id}_{M\otimes M},\ \sigma_{M,A})$.

Theorem (L.-W. 2015): GYD modules are many

✦ $\mathcal{Z}_A^C$ is strict monoidal:





✦ $\mathcal{Z}_A^C$ contains $(A, \sigma_{C,A}, \sigma_{A,A})$, $(C, \sigma_{C,C}, \sigma_{C,A})$, and all their mixed tensor products.

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Braided system $(\mathcal{C}, \mathcal{A}; \overline{\sigma}) \rightsquigarrow$ $\mathcal{Z}_{\mathcal{A}}^{\mathcal{C}}$ = the category of braided systems $(\mathcal{C},\ M,\ \mathcal{A};\ \overline{\sigma},\ \sigma_{\mathcal{C},M},\ \mathrm{Id}_{M\otimes M},\ \sigma_{M,\mathcal{A}})$.

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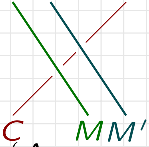
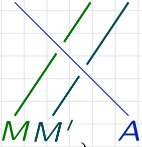
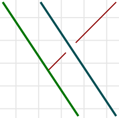
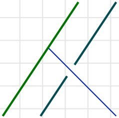




✦ $\mathcal{Z}_{\mathcal{A}}^{\mathcal{C}}$ contains $(\mathcal{A}, \sigma_{\mathcal{C},\mathcal{A}}, \sigma_{\mathcal{A},\mathcal{A}})$, $(\mathcal{C}, \sigma_{\mathcal{C},\mathcal{C}}, \sigma_{\mathcal{C},\mathcal{A}})$, and all their mixed tensor products.

✦ $$\boxed{E_{\mathcal{A}}^{\mathcal{C}} \colon \mathcal{YD}_{\mathcal{A}}^{\mathcal{C}} \times \mathcal{Z}_{\mathcal{A}}^{\mathcal{C}} \to \mathcal{YD}_{\mathcal{A}}^{\mathcal{C}}}$$





10 Categorical center

$$E_A^C : \mathcal{YD}_A^C \times \mathcal{Z}_A^C \to \mathcal{YD}_A^C$$

Example: ① Hopf algebra $(H, \mu, \nu, \varepsilon, \Delta, S)$

$\rightsquigarrow$ braided system $(A = H, C = H; \overline{\sigma})$ & $\pi = \mathrm{Id}_H$.

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$\rightsquigarrow$ two braidings for H:

$$\sigma_H(h \otimes h') = h'_{(1)} \otimes S(h'_{(2)}) h h'_{(3)},$$

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Woronowicz braidings!

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$$E^C_A\colon \mathcal{YD}^C_A \times \mathcal{Z}^C_A \to \mathcal{YD}^C_A$$

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Woronowicz braidings!

✦ A more complicated gYD structure on $\Bbbk$

$\rightsquigarrow$ Hennings braidings.

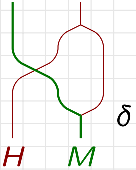
11 Structure on $\mathcal{YD}_A^C$

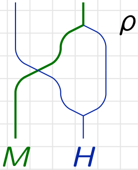
1. Hopf algebra $(H, \mu, \nu, \varepsilon, \Delta, S)$

$\rightsquigarrow$ braided system $(A = H, C = H; \overline{\sigma})$;

$\rightsquigarrow$ $E_A^C \colon \mathcal{YD}_A^C \times \mathcal{Z}_A^C \to \mathcal{YD}_A^C$.

Proposition (L.-W. 2015): $Z_{YD} \colon \mathcal{YD}_A^C \hookrightarrow \mathcal{Z}_A^C$





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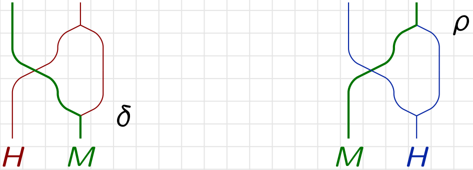

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3. The same for crossed modules of groups.

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6 Crossed module of shelves $(R, S, \pi, \cdot)$

$\rightsquigarrow$ braided system $(A = S, C = R; \overline{\sigma})$.

Proposition (L.-W. 2015): $Z_{SD}, \tilde{Z}_{SD} \colon \mathcal{YD}_A^C \hookrightarrow \mathcal{Z}_A^C$

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Theorem (L.-W. 2015):

- ✦ $\otimes$ is a pre-tensor structure.
- ✦ $\widetilde{\otimes}$ is a non-strict pre-tensor structure, with

interesting associators

$$\alpha_{M,M',M''} \colon (M \otimes M') \otimes M'' \xrightarrow{\sim} M \otimes (M' \otimes M''),$$

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Question: A braided pre-tensor structure?

11 Structure on $\mathcal{YD}_A^C$

5 Crossed module of Lie algebras $(L, N, \pi, [])$

$\rightsquigarrow$ braided system $(A = N^+, C = L^+; \overline{\sigma})$.

Proposition (L.-W. 2015): $\boxed{\mathcal{YD}_A^C \hookrightarrow \mathcal{Z}_A^C}$

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