Self- and multi-distributivity with a braided flavor

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I. Structural pre-braidings — II. Braided homology

algebraic structure + modules $\overset{\text{case by case}}{\rightsquigarrow}$ (pre-)braiding + braided module $\overset{\text{canonical}}{\longmapsto}$ (bi-)complex $\longmapsto$ homology & other info

① UAA (= unitary associative algebra) + algebra module $\rightsquigarrow$ $\sigma_{Ass} = \nu \otimes \mu : a \otimes b \mapsto 1 \otimes a \cdot b$ or $\sigma^{\tau}_{Ass} = \mu \otimes \nu : \ldots \mapsto a \cdot b \otimes 1$ $\longmapsto$
- bar c-x
- Hochschild c-x

② SD (= self-distributive) structure + rack-set $\rightsquigarrow$ $\sigma_{SD} : (a, b) \mapsto (b, a \triangleleft b)$ $\longmapsto$
- 1-term distributive c-x
- rack c-x
- quandle c-x
- twisted versions

③ Lie/Leibniz algebras

④ Bialgebras

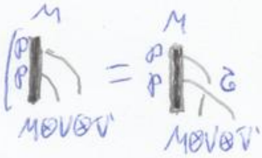
⑤ Hopf & Yetter-Drinfel'd (bi-)modules etc.

III. Some refinements

I. Structural pre-braidings.

We work in a strictly monoidal category $\mathcal{C}$ (often $Vect_{k}$, Mod_R, $Mod(gr_R)$).

Pre-braided object: $(V \in \mathcal{C}, \sigma : V \otimes V \circlearrowleft)$ + YBE.

Braided module over (V, σ): $(M \in \mathcal{C}, \rho : M \otimes V \to M)$ + $\Big($  $\Big)$

Braided character: braided module $(I, \epsilon : V \to I)$.

Rmk: left = right.

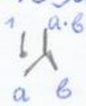
Categories: $Br(\mathcal{C})$, $Mod_{(V,\sigma)}$,

$Br^{\bullet}(\mathcal{C})$, $Mod^{\bullet}_{(V,\sigma,\gamma)}$ } "pointed"

with a chosen "element" $\gamma : I \to V$ — γ "acts by identity"

Prop.:

① • $\boxed{UAA(\mathcal{C}) \xrightarrow[\text{functor}]{\text{fully faithful}} Br^{\bullet}(\mathcal{C})}$

$(V, \mu, \nu) \longmapsto (V, \sigma_{Ass} = \nu \otimes \mu, \nu)$ 

$f \longmapsto f$

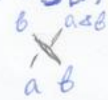
• associativity of $\mu \Longleftrightarrow$ YBE for σ_{Ass} (if ν is a unit)

• $Mod_V \simeq Mod^{\bullet}_{(V, \sigma_{Ass}, \nu)}$

$(M, \rho) \longleftrightarrow (M, \rho)$.

• highly non-invertible: $\sigma^2_{Ass} = \sigma_{Ass}$.

② • Shelf $\xrightarrow[\text{functor}]{\text{f.f.}}$ $Br(Set)$

$(S, \triangleleft) \longmapsto (S, \sigma_{SD})$ 

$f \longmapsto f$

• SD for $\triangleleft \Longleftrightarrow$ YBE for σ_{SD}

• $Mod_S \simeq Mod_{(S, \sigma_{SD})}$

$(s \cdot a) \cdot b = (s \cdot b) \cdot (a \triangleleft b)$

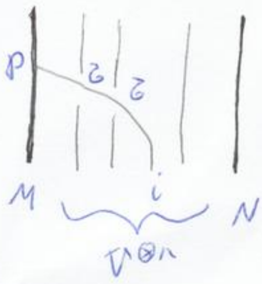
• rack cond^n $\Longleftrightarrow$ invertibility for σ_{SD}

II. Braided homology

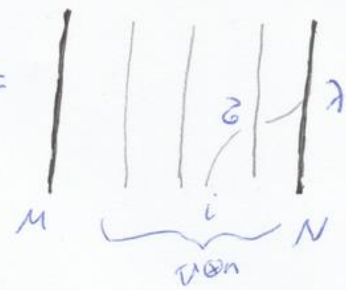
① **Th.:** In a strictly mon. pre-additive cat. $\mathcal{C}$, take:
- a $\overset{\text{pre-}}{\text{braided}}$ object (V, σ)
- a right module (M, ρ) } over (V, σ)
- a left — " — (N, λ) } over (V, σ)

Then $C_n := M \otimes V^{\otimes n} \otimes N$ can be endowed with a pre-bisimplicial structure

${}^{\rho}d_{n;i} :=$

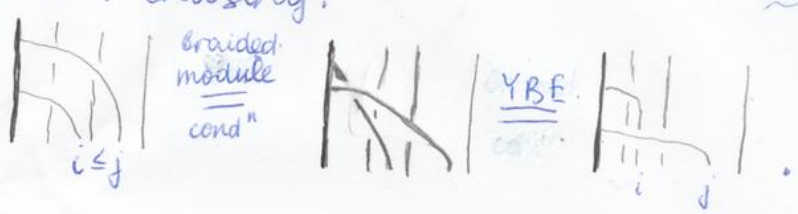


$d^{\lambda}_{n;i} :=$



Rmk: → Reasoning in terms of strands, one avoids the tiresome index-chasing.

Proof:

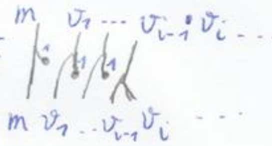


→ To take into account the $(-1)^{i-1}$ sign, replace σ with $-\sigma$ (i.e. count the crossings).

→ We need local (hence simpler) properties rather than the global ones (def^n of simplicial cat.

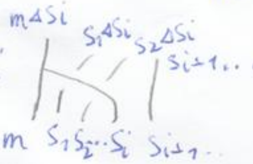
Crl: $\left({}^{\rho}d_n := \sum_{i=1}^{n} (-1)^{i-1}\, {}^{\rho}d_{n,i}\right),\ d^{\lambda}_n := \sum_{i=1}^{n} (-1)^{i-1} d^{\lambda}_{n,i}$) is a bidifferential on C_n.

Ex: ① UAA: ${}^{\rho}d_{n,i} =$



$\Rightarrow {}^{\rho}d =$ bar diff^e with coefficients on the left.

② SD:



$V = kS$ or k, S

$\Rightarrow {}^{\rho}d =$ 1-term distributive diff^e with coeff^s on the left.

${}^{\varepsilon}d - d^{\varepsilon} =$ rack diff^e

${}^{t\varepsilon}d - d^{\varepsilon} =$ twisted rack diff^e

$\varepsilon: a \mapsto 1 \ \forall a \in S$
↑ braided character

② Pre-braided coalgebra (V, σ, Δ) in $\mathcal{C}$:

→ pre-br. object (V, σ);

→ co-associative coalg. (V, Δ);

→ compatibilities: 

Semi-braided coalgebra: only ↑.

σ-cocommutativity: 

<u>Th. Bis</u>: • If moreover (V, σ, Δ) is a pre-braided coalgebra, then $(C_n, {}^{\rho}d_{n,i}, d^{\lambda}_{n,i}, \underline{S_{n,i}} := \underbrace{|\,|}_{M}\,\underset{i}{|\,Y_\Delta}\,\underbrace{|\,|}_{N})$ is a very weakly bisimplicial structure, becoming weakly simplicial if Δ is σ-cocommutative.

• If (V, σ, Δ) is only semi-braided, one should work with $(C_n, {}^{\rho}d_{n,i}, S_{n,i})$.

<u>Crl</u>: $\underbrace{\sum_{i=1}^{n} Im(S_{n,i})}_{=: D_{n+1}}$ is a sub-bicomplex, called degenerate.

<u>Ex</u>: ① UAA: $\Delta_1 : a \mapsto 1 \otimes a$.

<u>Lemma</u>: $(V, \sigma_{Ass}, \Delta_1)$ is a pre-br. σ-cocomm. coalgebra.

$D_n := Span\{ v_1 \otimes \ldots \otimes v_{i-1} \otimes 1 \otimes v_{i+1} \ldots \otimes v_n \mid 1 \le i \le n-1 \}$; $(C_n, {}^{\rho}d_n - d^{\lambda}_n)/D_n \rightsquigarrow$ Hochschild c-x

② SD. $\Delta_D : a \mapsto a \otimes a$.

<u>Lemma</u>: → semi pre-braided

→ pre-braided ⇔ $a \lhd b = (a \lhd b) \lhd b \quad \forall a, b \in S$

→ σ-cocomm. ⇔ S is a spindle: $a \lhd a = a \quad \forall a \in S$.

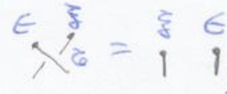
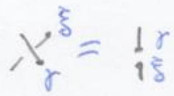
$D_n := Span\{ (a_1, \ldots, a_{i-1}, a_i, a_i, a_{i+1}, \ldots, a_{n-1}) \mid 1 \le i \le n-1 \}$; $(C_n, {}^{\varepsilon}d_n - d^{\varepsilon}_n)/D_n \rightsquigarrow$ quandle c-x

③ Concatenation: $h_\gamma :=$ 

Arrow operation: ${}^{\epsilon}\pi_\gamma :=$ 

$\underline{Ex}$.: ① $v_1 \otimes \dots \otimes v_n \longmapsto v_1 \otimes \dots \otimes v_{n-1} \otimes v_n \cdot w$

② $(a_1, \dots, a_n) \longmapsto \epsilon(b)(a_1 \triangleleft b, \dots, a_n \triangleleft b)$.

Prop.:
- ${}^{\xi}d \circ {}^{\epsilon}\pi_\gamma = {}^{\epsilon}\pi_\gamma \circ {}^{\xi}d$ if  $(*)$
- $d^{\xi} \circ {}^{\epsilon}\pi_\gamma = {}^{\epsilon}\pi_\gamma \circ d^{\xi}$ if  $(**)$
- ${}^{\xi}d \circ h_\gamma = h_\gamma \circ {}^{\xi}d + (-1)^n \, {}^{\xi}\pi_\gamma$
- $d^{\xi} \circ h_\gamma = h_\gamma \circ {}^{\xi}d + (-1)^n \, {}^{\xi}_{\gamma} \cdot \mathrm{Id}$ if $(**)$
- ${}^{\epsilon}\pi_\gamma = {}^{\epsilon}_{\gamma} \, \mathrm{Id}$ if  $(***)$

$\underline{Ex}$.: ② $\xi(a \triangleleft b) = \xi(a) \ \forall a, b$ s.t. $\epsilon(b) \neq 0$
(here R has no zero devisors),
e.g. $\rightarrow \xi : a \mapsto 1 \ \forall a \in S$
$\rightarrow a \triangleleft b = a \ \forall a, b \in S$
$\rightarrow \epsilon = \xi = \delta_{a,x}$: $a \triangleleft x = x \Leftrightarrow a = x$
fixed — o.k. if S is a quandle

$(**)$ $\gamma : \overset{\in R}{\alpha} \mapsto \alpha \cdot c$, $c \in S$
$\xi(a \triangleleft c) = \xi(a) \ \forall a \in S$

$(***)$ $a \triangleleft c = a \ \forall a \in S$

III. Some refinements

① Pre-Braided system: $V_1, \dots, V_r$; $\sigma_{i,j}: V_i \otimes V_j \to V_j \otimes V_i$ $\forall \underline{i \le j}$;
$+$ YBE on $V_i \otimes V_j \otimes V_k$ $\forall \underline{i \le j \le k}$.

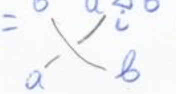
$\leadsto$ Bialgebras, Hopf & YD (bi) modules etc.

$\leadsto$ multi-braided tensor products of algebras

② Multi-Braided object: $(V; \sigma_1, \dots, \sigma_r : V \otimes V \circlearrowleft)$ + mixed YBE:

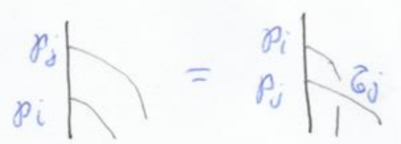
$\forall 1 \le i, j \le r$.

Ex.: ② a set S, $\sigma_i =$



; all mixed YBE $\Leftrightarrow$ multi-distributivity.

Multi-Braided module: $(M; \rho_i : M \otimes V \to M, 1 \le i \le r)$ +



$\forall i, j$.

Cf. $(*)$ for the case $M = I$.

Ex.: ② partial derivatives $\delta_{0,x}d$.

Th.^{multi}: • $(M \otimes V^{\otimes n}, \rho^1 d, \dots, \rho^r d)$ is a differential multi-complex.

• pre-multisimplicial & weakly multisimplicial structure.

Ex.: ② multi-term distributive differential.