

Groupes de tresses & groupes libres

Victoria Lebed,
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- Plan:
- B_n) Tresses : l'art de la coiffure
 - P_n) Tresses pures (et dures)
 - ↳) Les tresses passent à l'action
 - τ) Calcul : Johnson | tresses
 - III) Algèbres de battage quantiques

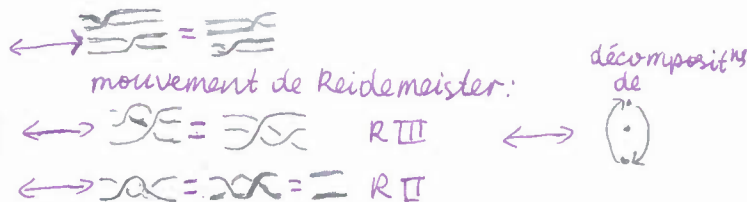
B_n) Tresses et leurs diverses avatars

① $\mathcal{B}_n =$ gpe des tresses géométriques à n brins / isotopie
opération: concaténation + contraction

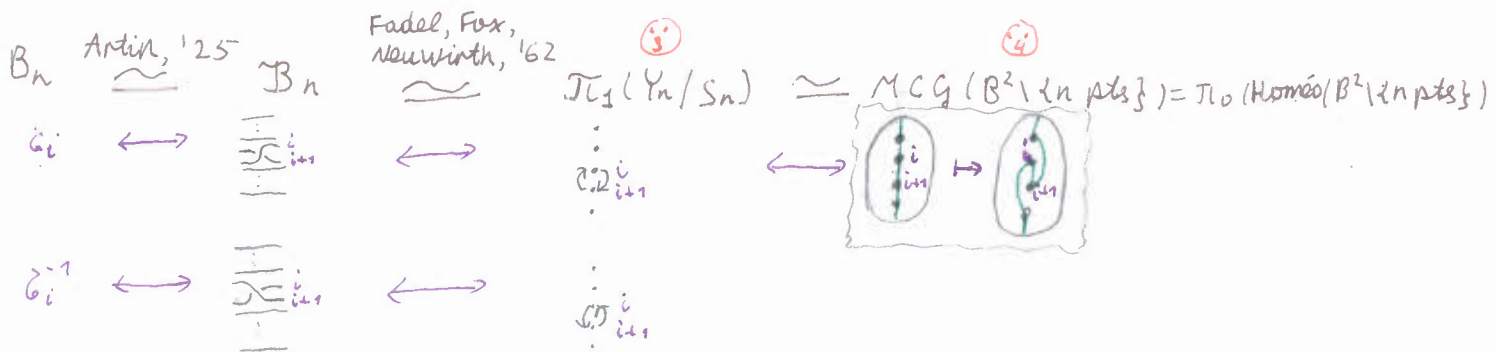


$$\begin{aligned} \boxed{B \circ B'} &= \boxed{B} \boxed{B'} \\ \boxed{1} &= \equiv \\ \boxed{B^{-1}} &= \equiv \boxed{B} \text{ (image miroir)} \end{aligned}$$

② $B_n = \langle \delta_1, \dots, \delta_{n-1} \rangle / \delta_i \delta_j = \delta_j \delta_i, |i-j| > 1$
 $\delta_i \delta_{i+1} \delta_i = \delta_{i+1} \delta_i \delta_{i+1}$ (Yang-Baxter)
 $\delta_i \delta_i^{-1} = \delta_i^{-1} \delta_i = 1$



Rmq! Un système de générateurs alternatif: $(z_1, \delta_1 \delta_2 \dots \delta_{n-1})$.



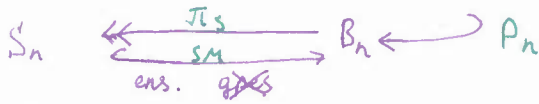
$$S_n \curvearrowright Y_n = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j\}$$

Rmq! On peut remplacer $\mathbb{C} \simeq \mathbb{R}^2$ et B^2 par d'autres surfaces \leadsto généralisation des tresses.

P_n) Tresses pures

$$S_n = \text{Perm}(\{1, 2, \dots, n\}) \simeq B_n / \langle \sigma_i^2 = 1 \rangle$$

$$s_i = (i, i+1) \longleftarrow \sigma_i$$



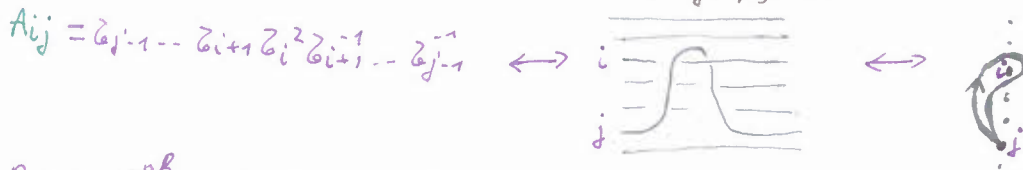
Rmq: $P_n \simeq \pi_1(Y_n)$

$$s_i \dots s_k \longleftarrow \sigma_{i_1} \dots \sigma_{i_k}$$

(une d\u00e9compositⁿ min.
en transpositions
\u00e9l\u00e9mentaires)

s_m : section de Matsumoto

Burau, '32: $P_n \simeq \langle A_{ij} \mid 1 \leq i < j \leq n \rangle / A_{rs}^{-1} A_{ij} A_{rs} = \dots$

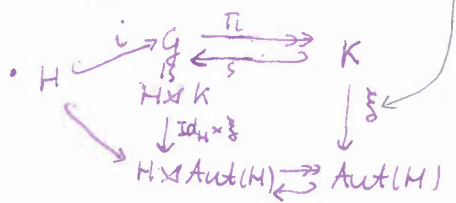


Rmq: $P_n^{ab} \simeq \bigoplus_{1 \leq i < j \leq n} \mathbb{Z} A_{ij} \simeq \mathbb{Z}^{\frac{n(n-1)}{2}}$

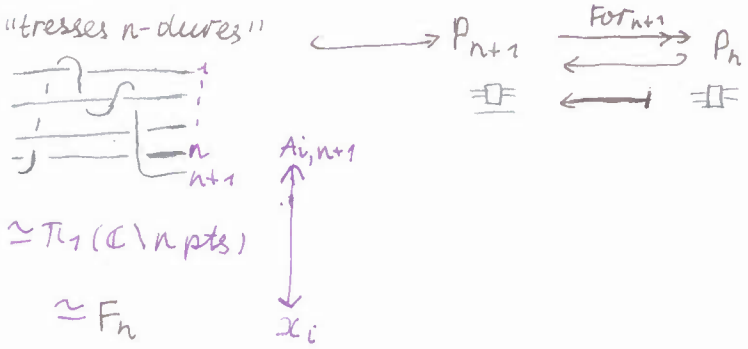
Lemme 1: Soit $M \xrightarrow{i} G \xrightleftharpoons[s]{\pi} K$ une suite exacte scind\u00e9e de g\u00e9s. Alors:

- $K \triangleleft H$, $k \cdot h = i^{-1}(s(k)i(h)s(k)^{-1})$

- $G \xrightleftharpoons[\text{is}]{\text{is}} H \times K$

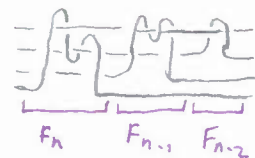


Fr\u00f6hlich, '36:



Lemme 1

$$\implies P_{n+1} \simeq F_n \times P_n \simeq \dots \simeq F_n \times (F_{n-1} \times \dots \times F_1)$$



forme coiff\u00e9e,
ou normale
(Markov, '45;
Artin, '47)

\(\rightarrow\) filtration normale

$$1 = P_n^{(0)} \subset P_n^{(1)} \subset \dots \subset P_n^{(n-1)} = P_n$$

$$P_n^{(i)} \hookrightarrow P_n \xrightarrow{\text{For}_{i, n-i}} P_{n-i}$$

$$\text{avec } P_n^{(i)} / P_n^{(i-1)} \simeq F_{n-i}$$

2) Action des groupes de tresses

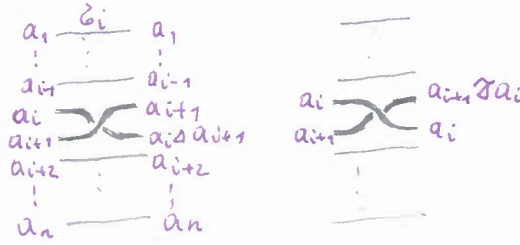
Rack: (R, \triangleleft) t. q.

1) $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$

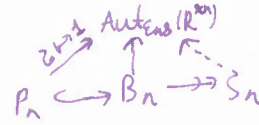
2) $\exists \tau$ t. q. $(a \triangleleft b) \tau b = a = (a \tau b) \triangleleft b$

$\left\{ \begin{array}{l} \Leftrightarrow R_{III} \\ \Leftrightarrow R_{II} \end{array} \right.$

Lemme 2: $R^n \triangleleft B_n$



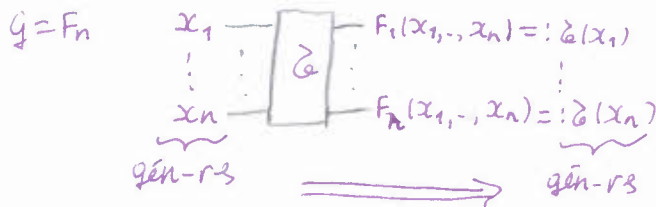
Exemples: • rack trivial: R q.c.q., $a \triangleleft b = a$



• rack d'Alexander: $M \in \mathbb{Z}[t^{\pm 1}] \text{ Mod}$, $a \triangleleft b = ta + (1-t)b$

$\leadsto M^{\oplus n} \triangleleft B_n, B_n \xrightarrow{PA} GL_n(\mathbb{Z}[t^{\pm 1}])$ (rep. de Burau)

• rack de conjugaison: gpe G , $a \triangleleft b = b^{-1}ab$



$\leadsto B_n \triangleleft F_n$ (rep. d'Artin; Hurwitz, 1891)

Artin, '25: • $B_n \xrightarrow{PA} \text{Aut}(F_n)$

• $\text{Im } PA = \left\{ \varphi \in \text{Aut}(F_n) \mid \begin{array}{l} \bullet \varphi(x_1 \dots x_n) = x_1 \dots x_n \\ \bullet \exists M \in S_n \text{ t. q. } \forall i, \varphi(x_i) \stackrel{\text{conj.}}{\sim} x_{M(i)} \end{array} \right\}$

5)

url: Solution pour le pb du mot dans B_n .

calcul:



Interprétations topologiques de PA:

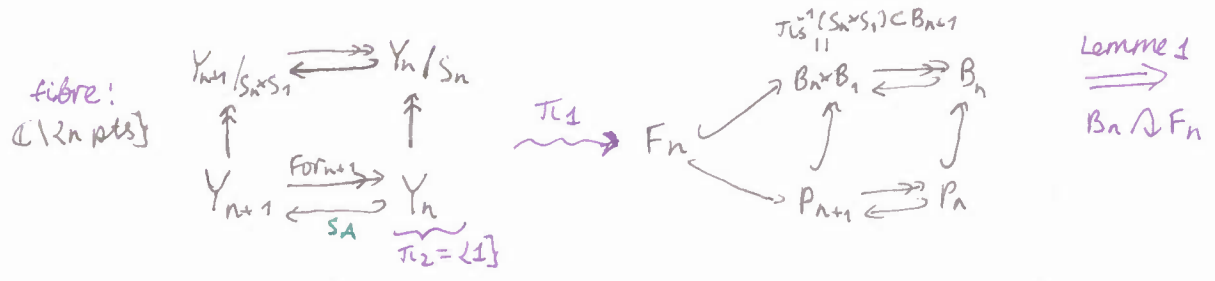
(1) $B_n \rightarrow \text{MCG}(B^2 \setminus \{n \text{ pts}\}) \hookrightarrow \pi_1(B^2 \setminus \{n \text{ pts}\}; p) \simeq F_n$



(2) $\pi_1(\mathbb{R}^2 \setminus \{n \text{ pts}\}) \xrightarrow{\cong} \pi_1(\mathbb{R}^2 \times [0,1] | \mathbb{R}^2) \xleftarrow{\cong} \pi_1(\mathbb{R}^2 \setminus \{n \text{ pts}\})$

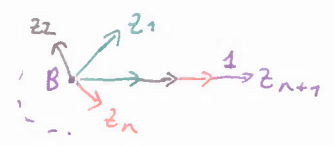
$\downarrow \cong$ $\downarrow \cong$ $\downarrow \cong$
 F_n F_n F_n
 \downarrow \downarrow \downarrow
 $\text{PA}(\mathbb{R}^2)$

(3) fibrations localement triviales



$(z_1, \dots, z_n, B + \sum_j |z_j - B| + 1) \xleftarrow{SA} (z_1, \dots, z_n)$ (Arnol'd, 69)

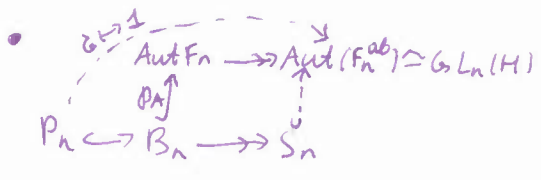
$B = \frac{1}{n} \sum z_i$



τ) L'application de Johnson τ_1^{std} restreinte aux types de tresses

• $F_n \xrightarrow{ab.} F_n^{ab} =: H \simeq \bigoplus_{i=1}^n \mathbb{Z} X_i$
 $x_i \mapsto X_i$
 notatⁿ: $\sigma \mapsto [\sigma]$

$H^* := \text{Hom}_{\mathbb{Z}}(H, \mathbb{Z}) \simeq \bigoplus_{j=1}^n \mathbb{Z} e_j$
 $\langle e_j, X_i \rangle = \delta_{i,j}$



notatⁿ:
 $|\sigma| \in GL_n(H) \text{ ou } GL_n(H^*)$
 \uparrow
 $\sigma \in S_n \text{ ou } B_n$

explicitement, $\forall s \in S'$,
 $s \cdot X_i = X_{s^{-1}(i)}$,
 $s \cdot e_j = e_{s^{-1}(j)}$
 $\square \langle s \cdot e_j, X_i \rangle = \langle e_j, s^{-1} X_i \rangle = \langle e_j, X_{s(i)} \rangle$
 $= \delta_{j, s(i)} = \delta_{s^{-1}(j), i} = \langle e_{s^{-1}(j)}, X_i \rangle$

• $std: F_n \xrightarrow{\text{gpe}} \mathbb{Z} \langle X_1, \dots, X_n \rangle$
 $x_i \mapsto 1 + X_i$
 $x_i^{-1} \mapsto 1 - X_i + X_i^2 - X_i^3 \dots$

$\leadsto std_2: F_n \xrightarrow{\text{gpe}} \bigoplus_{1 \leq i, j \leq n} \mathbb{Z} X_i X_j \simeq M^{\otimes 2}$

Lemme 3: $std_2(x_{i_1}^{\epsilon_{i_1}} \dots x_{i_k}^{\epsilon_{i_k}}) = \sum_{\alpha < \beta} \epsilon_{i_\alpha} \epsilon_{i_\beta} X_{i_\alpha} X_{i_\beta} + \sum_{\epsilon_{i_\alpha} = -1} X_{i_\alpha}^2$

Cas part.: $std_2(x_i) = 0$

• $std_2(x_i x_j x_i^{-1}) = X_i X_j - X_j X_i$
 • $std_2(x_i x_j x_i^{-1} x_j) = X_i X_j - X_j X_i$

• $\tau_1 = \tau_1^{std} |_{B_n}: B_n \hookrightarrow \text{Aut } F_n \rightarrow \text{Hom}_{\mathbb{Z}}(H, M^{\otimes 2}) \simeq M^* \otimes M^{\otimes 2}$
 \downarrow
 $\sigma \mapsto ([\sigma]) \mapsto std_2(\sigma) - (\sigma \cdot std_2)(\sigma)$
 \uparrow
 F_n
 $= | \sigma |^{\otimes 2} std_2(\sigma^{-1} \sigma)$

Lemme 4: τ_1 est bien définie
 (rappel) $\tau_1(1) = 0$

• $\tau_1(\sigma \sigma') = \tau_1(\sigma) + |\sigma| \cdot \tau_1(\sigma')$ (i.e., τ_1 est un 1-cocycle)

• Lemme 5 (K., 2.1): $\tau_1(\delta_i) = e_i \otimes (X_i X_{i+1} - X_{i+1} X_i)$.

$\square \tau_1(\delta_i): X_j = [X_j] \mapsto \underbrace{std_2(x_j)}_{=0} - |\delta_i|^{\otimes 2} std_2(\delta_i^{-1} x_j) \stackrel{L3}{=} -\delta_{i,j} |S_i|^{\otimes 2} (X_i X_{i+1} - X_{i+1} X_i) = \delta_{i,j} (X_i X_{i+1} - X_{i+1} X_i)$

• Lemme 6 (K., 2.2): $\tau_1(A_{ij}) = (e_i - e_j) \otimes (X_i X_j - X_j X_i)$

$\square A_{ij} = \sigma \delta_i^2 \sigma^{-1}, \sigma = \delta_{j-1} \delta_{j-2} \dots \delta_{i+1}$; $\rightarrow \pi_S(\delta_i^2) = 1 \in S_n$
 $\rightarrow \pi_S(\sigma) = (i, i+1, \dots, j) =: \tilde{\sigma}$
 $\rightarrow |\delta_i| = |\pi_S(\delta_i)|$

$\tau_1(\sigma \delta_i^2 \sigma^{-1}) \stackrel{L4}{=} \tau_1(\sigma) + |\sigma| \tau_1(\delta_i^2) + |\sigma \delta_i^2| \tau_1(\sigma^{-1}) \stackrel{L4}{=} \tau_1(\sigma \cdot \sigma^{-1}) + |\tilde{\sigma}| (\tau_1(\delta_i) + |S_i| \tau_1(\delta_i)) \stackrel{L5}{=} 0 + |\tilde{\sigma}| (e_i \otimes (X_i X_{i+1} - X_{i+1} X_i) + |S_i| (e_i - e_{i+1}) \otimes (X_i X_{i+1} - X_{i+1} X_i))$

$= |\tilde{\sigma}| (e_i \otimes (X_i X_{i+1} - X_{i+1} X_i) + e_{i+1} \otimes (X_{i+1} X_i - X_i X_{i+1})) = [\tilde{\sigma}^{-1}(i) = j, \tilde{\sigma}^{-1}(i+1) = i] = (e_i - e_{i+1}) \otimes (X_i X_j - X_j X_i) \quad \square$

III) Algèbres de Battage quantiques

Soit $V \in_A \text{Mod}$ et $\mathcal{Z}: V \otimes V \rightarrow V \otimes V$ t. q. $B_n \xrightarrow{\text{q.t.e.}} \text{Aut}_A(V^{\otimes n})$
 $\mathcal{Z}_i \mapsto \text{Id}_V^{\otimes i-1} \otimes \mathcal{Z} \otimes \text{Id}_V^{\otimes n-i-1}$

Rmq: $\cdot \Leftrightarrow$ Yang-Baxter pour \mathcal{Z}

B_n^+ monöide $\rightarrow \text{End}_A(V^{\otimes n})$ suffit
 } monöide de tresses positives à n brins

Ex.: rack $R \rightsquigarrow V = AR$, $\mathcal{Z}: (a, b) \mapsto (b, a \triangleleft b)$ (+ linéarisation).

$Sh_{p,q} := \{ s \in S_{p+q} \mid \begin{matrix} s(1) < s(2) < \dots < s(p) \\ s(p+1) < \dots < s(p+q) \end{matrix} \}$ (battages)

$\mathbb{Z}\langle \mathcal{Z} \rangle_{p,q} := \sum_{s \in Sh_{p,q}} \rho^{\mathcal{Z}}(s_M(s)) \in \text{Hom}_A(V^{\otimes p+q}, V^{\otimes p+q})$

produit de battage quantique

\mathcal{Z} général vs. $\mathcal{Z}: v \otimes w \mapsto w \otimes v$



Rosso, '95: $(T(V), \mathbb{Z}\langle \mathcal{Z} \rangle)$ est une algèbre associative unitaire;
 • algèbre de Hopf tressée;
 • \mathcal{Z} -commutative si $\mathcal{Z}^2 = \text{Id}$.

Dans la construction du morphisme de Gysin, une certaine transformation $C^n(G, M) \rightarrow C^n(G, M)$, $g \mapsto g_j$, joue un rôle important (cf. l'exposé de Friedrich, ou Hochschild-Serre). En fait, elle s'exprime facilement en termes du produit de battage quantique:

$$g_j = g \circ \mathbb{Z}\langle \mathcal{Z} \rangle_{n-1,1}^{-1}, \text{ où } \mathcal{Z}: \mathbb{Z}G^{\otimes 2} \rightarrow \mathbb{Z}G^{\otimes 2}$$

$$(a, b) \mapsto (b, b^{-1}ab).$$

Ex.: $g_{n-1}(a, B_1, \dots, B_{n-1}) = \sum_{i=1}^n (-1)^{i-1} g(B_1, \dots, B_{i-1}, (B_1 \dots B_{i-1})^{-1} a B_1 \dots B_{i-1}, B_{i+1}, \dots, B_{n-1})$.