ITERATES OF FRACTIONAL ORDER

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1. Introduction. The body of this paper is a complete answer to the following question:

Let E be any space whatever. $g(x)$ is a function[1] mapping E into E. When does there exist a function $f(x)$, of the same type, such that

$$f(f(x)) = g(x) \qquad (x \in E)? \tag{1}$$

This problem typifies the general one of iteration. Let $g^k(x)$ be the kth order iterate of g [i.e. $g^0(x) = x$, $g^{k+1}(x) = g(g^k(x))$]. The iteration problem is that of attaching a consistent meaning to this expression for fractional k (in the sense of preserving the additive law of exponents). An f satisfying (1) is thus $g^{1/2}(x)$. By ideas similar to those discussed herein, we can find the most general $g^{1/m}$ and then by iterating it, the most general iterate of any rational order. Without introducing continuity, this is as far as it is possible to go. We confine ourselves to the case of $k = 1/2$ to avoid oppressive detail; the generalization to $k = 1/m$ is indicated later.

The iteration problem has received attention for many years, alone or as part of another topic (functional equations, fractional derivatives, the tri-operational algebra of Menger [1], etc.). Some of these applications require subsidiary conditions on the functions (continuity, differentiability, etc.). We deal with the general problem without such side conditions; thus our work might be called combinatorial. The problem with a side condition such as continuity appears highly interesting.

In all the literature we have encountered, the general problem is approached in but one way—through the Abel function. The idea here is to ascertain a numerically valued function ϕ on E satisfying

$$\phi(g(x)) = \phi(x) + 1.$$

Then iterates of all orders are obtained at once by

$$g^k(x) = \phi^{-1}(\phi(x) + k).$$

We show later that in a widespread class of cases, a ϕ does not exist. Even when it does, its inverse may not exist. Yet iterates of some or all fractional orders may exist. The non-existence of ϕ may hold even when we have continuity with respect to both x and k, as we shall show below.

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[1] If g is not defined for all of E, it suffices that our later criterion hold for some extension of g which is. If the range and domain of g are distinct we can thus take E to be their union.

For the Abel function approach in the complex number domain see the papers by Schwarzschild, Chayoth, and Koenigs [2]. For the real domain, Lyche [3] gives existence conditions for ϕ and continuous ϕ by methods somewhat akin to ours. Bödewadt [4] treats the case of a fully differentiable ϕ (real domain). Hadamard [5] summarizes two recent contributions.

Our interest in this question arose from the following problem propounded by Menger. Let E be R_1 and $g(x) = a + bx$. There is obviously a linear solution to (1) when $b \geqslant 0$, namely $f(x) = a/(1 + b^{\frac{1}{2}}) + b^{\frac{1}{2}}x$. Do solutions exist when $b < 0$? The question is answered below.

The text will be clearer if we outline our method first.

An *orbit* (defined precisely later) is a subset of E whose elements are linked by the operation g. We can represent one graphically as in Figure 1 where the dots represent elements of E and the arrows show the course of g.

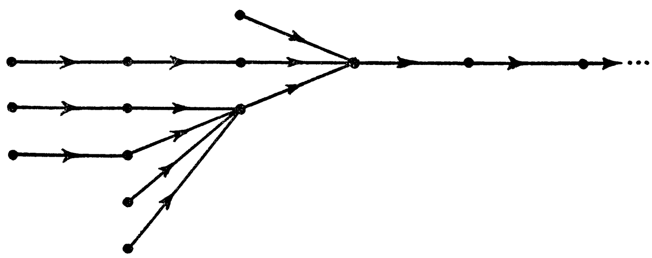

FIGURE 1

The present idea consists of constructing the orbits of f from those of g. Thus in Figure 2 we see two orbits with respect to g united so as to give one with respect to f. The dashed arrows show the course of f; the truth of (1) may be verified by noting that following two consecutive dashed arrows is equivalent to following one solid one.

This kind of construction can sometimes be carried out utilizing only a single g-orbit as in Figure 3.

We will show (Theorem 1) that these two instances typify the most general situation possible. The problem then reduces essentially to two questions: *When can two distinct g-orbits be "mated" (as in Figure 2) to produce one for f? When can a single g-orbit also be an f-orbit (as in Figure 3)?* which are answered by Theorems 3 and 4.

2. Orbits. Consider the following relation between the members x,y of E: There exist non-negative integers m, n such that

$$g^m(x) = g^n(y). \tag{2}$$

This relation is rst[2]; the classes into which it divides E, in the customary way, are called *orbits*.[3] The orbit containing x will be denoted by $L[x;g]$.

[2]Reflexive, symmetric and transitive.

[3]This concept appears in Lyche [3], where he attributes it to a suggestion of Kuratowski. He uses the term *class*; in a previous abstract of this work we used *linkage*. The term *orbit* appears in Whyburn [6].

A set

(3) $$x_1, \ldots, x_n$$

such that $g(x_1) = x_2, \ldots, g(x_n) = x_1$ will be called a *cycle* (*n-cycle*).

LEMMA 1. *An orbit contains at most one cycle.*

For let x and y be elements which belong to the same orbit but to two distinct cycles; (2) holds. The element on both its sides belongs to both cycles. The cycles, having a common element, are identical.

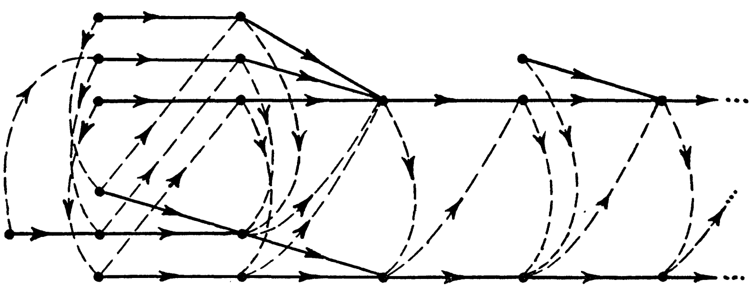

FIGURE 2

An orbit containing a cycle (of n elements) will be called *cyclic* (*n-cyclic*). Let C be the cycle of a cyclic orbit L. An element x_0 will be called a *leader* if

$$x_0 \in L - C, \quad g(x_0) \in C.$$

For a particular leader x_0 the subset of all y of L such that for some non-negative integer n

(4) $$g^n(y) = x_0$$

will be called a *branch* or more precisely a *branch from* $g(x_0)$.

LEMMA 2. *The branches constitute an aliquot, disjoint subdivision of* $L - C$. *For each* y, *the* n *of* (4) *is unique.*

Let $y \in L - C$; (2) holds for y and any x which $\in C$. Let n_1 be the smallest n such that $g^n(y) \in C$; $n_1 > 0$. Put $n = n_1 - 1$. Then $g^n(y) = x_0$ is a leader. This x_0 and n are unique, for suppose the existence of a second pair, i.e.

$$g^{n'}(y) = x'_0$$

and say $n \geqslant n'$. Then $x_0 = g^n(y) = g^{n-n'}(x'_0)$. Now $n - n' > 0$ is impossible as this implies $x_0 \in C$. Then $n = n'$, $x_0 = x'_0$.

Any union of branches from the same $z \in C$ is called a *branch cluster* (*from* z).

The subset of all y of a branch B for which the n of (4) is even (odd) is called the *even* (*odd*) *part of* B.

The following two operations concern only the structural properties of orbits, i.e. those invariant under isomorphisms (the term is used in the expected sense of a biunique, g-preserving correspondence). In other words, we admit orbits whose elements are abstract.

Consider the subsets X of an orbit L which are inverse images of single elements of E under g. (X is the set of $x \in L$ such that $g(x) = y$ for some fixed $y \in L$.) Divide each such X into a system of aliquot, disjoint subsets X_α. Identify the elements of each X_α into a single element, thus obtaining a new orbit L'. For L', g is defined by $g(X_\alpha) = g(x)$ where $x \in X_\alpha$; if $g(y) = x \in X_\alpha$ then for L', $g(y) = X_\alpha$. L' will be called a *contraction* of L. For

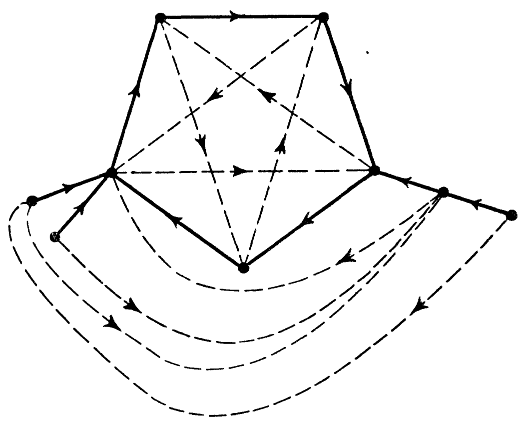

FIGURE 3

a cyclic orbit we may apply the idea to its branches. We include the possibility of contracting a branch cluster into a branch by identifying all its leaders.

By a *curtailment* of an orbit or branch L is meant the new orbit or branch arising when some of the elements x of L for which there is no y such that $g(y) = x$ are removed from L. (For the unremoved elements, g is unchanged.)

3. The existence conditions.

THEOREM 1. *If f, g satisfy* (1), *each orbit with respect to f is the union of two* (*possibly identical*) *orbits with respect to g. More precisely*:

$$L(x;f) = L(x;g) \cup L(f(x);g). \qquad (x \in E) \tag{5}$$

Let $y \in L(x;f)$. Then, for suitable m, n,

$$f^m(y) = f^n(x) \tag{6}$$

and also

$$f^{m+1}(y) = f^{n+1}(x). \tag{7}$$

One of (6), (7) has an even superscript on the left; let it be

$$f^{2k}(y) = f^{2j+\epsilon}(x), \qquad \epsilon = 0 \text{ or } 1$$

which can also be written

$$g^k(y) = g^j(f^\epsilon(x))$$

so that $y \in$ the right side of (5).

On the other hand, if $y \in L(x; g)$ or $L(f(x); g)$, (2) can be written

$$f^{2n}(y) = f^{2m}(x) \text{ or } f^{2m+1}(x).$$

Two distinct orbits capable of being paired together in the manner mentioned in Theorem 1 are said to be *mateable*. An orbit capable of being paired with itself will be said to be *self-mateable*.

The existence criterion for f is now clear.

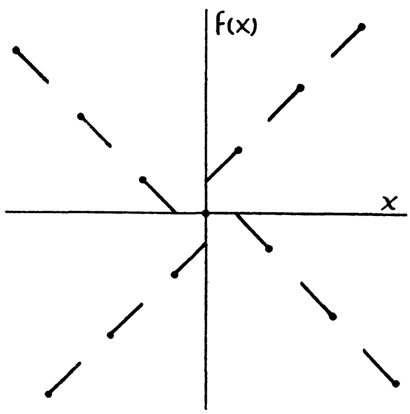

FIGURE 4

THEOREM 2. *A necessary and sufficient condition for f to exist is that the set of orbits with respect to g can be divided into three aliquot, disjoint subsets, such that two can be put into a biunique correspondence with mateable correspondents, while the third consists of self-mateable orbits.*

It remains to find criteria for mateability and self-mateability.

THEOREM 3. *A necessary and sufficient condition for two distinct orbits to be mateable is that a contraction of one be isomorphic to a curtailment of the other.*

Sufficiency. Let L_1, L_2 be orbits such that a contraction of L_1 is isomorphic to a curtailment L'_2 of L_2. If $x \in L_1$ then a subset of L_1, containing x, is paired by the isomorphism to $y \in L'_2$; define $f(x) = y$ and $f(y) = g(x)$. If $y \in L_2 - L'_2$, let $f(y)$ be any element of L_1 which is mapped by f into $g(y)$ (possible, as $g(y) \in L'_2$). The so-defined f satisfies (1).

Necessity. Let L_1, L_2 be the orbits. Identifying the x of L_1 for which $f(x)$ is the same element of L_2 gives a contraction L'_1 of L_1. Then f establishes an isomorphism between L'_1 and a subset of L_2. The excluded elements of L_2 may be removed by a curtailment.

Since the presence of an n-cycle is invariant under contractions and curtailments we have

COROLLARY. *n-cyclic orbits are mateable only with n-cyclic orbits.*

THEOREM 4. *A necessary and sufficient condition for an orbit L to be self-mateable is*

1) *L is n-cyclic with n odd. Let $n = 2k + 1$.*

2) *The branches of L are disjointedly the union of a set of branches S and a set of branch clusters $\overline{S}$. The S and $\overline{S}$ are in a biunique correspondence such that if $B \in S$ and $\overline{B} \in \overline{S}$ correspond, then a contraction of $\overline{B}$ is isomorphic to a curtailment of B and*[4] *if B is from z, $\overline{B}$ is from $g^k(z)$.*

Necessity. Let $x \in L$. As $f(x) \in L$, for suitable p, q,

$$g^p(x) = g^q(f(x))$$

or

$$f^{2p}(x) = f^{2q+1}(x). \tag{8}$$

As the two superscripts are distinct, familiar reasoning shows that for some j, $f^j(x)$ belongs to a cycle. Let it be C of order n. Let (3) be its elements so numbered that[5] $f(x_j) = x_{j+1}$. Then[5] $g(x_j) = x_{j+2}$. If n were even, the subsets of (3) with odd and even subscripts would each constitute a distinct cycle of L with respect to g. Put $n = 2k + 1$.

If $x \in C$, then $f(x) = f^{2k+2}(x) = g^{k+1}(x)$.

Now let B' be a branch with respect to f; x_0, its leader; B and $\overline{B}$, its even and odd parts. As $x_0 \in B$, B is not vacuous.

Letting $y \in B$, we must have for some $m \geqslant 0$

$$f^{2m}(y) = x_0 = g^m(y).$$

As x_0 is a leader with respect to g also, we see that, in regard to g, B is a branch from $g(x_0) = z$.

Similarly, if $y \in \overline{B}$,

$$f^{2m+1}(y) = x_0 = g^m(f(y))$$

which implies

$$g^{m+1}(y) = f(x_0) \in C.$$

Thus, in regard to g, $\overline{B}$ is a branch cluster from

$$f(x_0) = f^{2k+1}(f(x_0)) = g^{k+1}(x_0) = g^k(z).$$

Thus we have supplied the correspondence mentioned in 2). That a contraction of $\overline{B}$ is isomorphic to a curtailment of B follows as in the proof of Theorem 3.

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[4]We admit vacuous branch clusters, but not vacuous branches.

[5]Reckoned mod n.

Sufficiency. If x is in the cycle of the given orbit we define:

$$f(x) = g^{k+1}(x).$$

Now let B and $\bar{B}$ be as in 2). An f can be defined for their members as in the proof of Theorem 3, with evident modifications.

4. Inadequacy of the Abel function method. Lyche has shown that (in the case of functions of a real variable, but the result is true generally):

A necessary and sufficient condition for the Abel function to exist is that for no positive integer n and $x \in E$ is $g^n(x) = x$.

In other words, the condition is that there be no cyclic orbits. Our conditions show that f may exist in the contrary case. For example, let the orbit diagrammed in Figure 3 comprise the entire space E.

The truth of a fixed point theorem is equivalent to the existence of a 1-orbit. Thus the non-existence of the Abel function is not uncommon.

Now let E be the set of all non-negative numbers and $g(x) = x^2$. If we define $g^k(x)$ to be x^{2^k} we have a consistent iterate for each real k. Yet ϕ does not exist as 0 and 1 each belong to a 1-cycle.

We can easily construct the Abel function using the diagrams of non-cyclic orbits. In Figure 1, say, assign a real number to each vertical bank of dots in such a way that these numbers increase by unity as we proceed to the right. Doing this for each orbit (assumed non-cyclic), we obtain the most general Abel function. The truth of Lyche's theorem now becomes apparent.

For an Abel function to have an inverse it is clearly necessary that each vertical bank contain at most one dot. Further, the numbers must be assigned so as to avoid duplication of values on different orbits. If an Abel function is to be usuable for constructing iterates of all real orders, there must be a large enough number of orbits for each real number to occur once among its function values.

5. Examples: *The Menger Problem.* Let E be R_1 and $g(x) = a + bx$. If $b < 0$, our technique enables us still to construct solutions of (1), but they will never be continuous.

To illustrate, we take the case: $g(x) = -x$. Here, the orbit containing 0 is a 1-cycle. All other orbits are 2-cycles containing x and $-x\,(x \neq 0)$; there is thus exactly one containing a given positive number. The former can and must be self-mated; the latter are mateable in pairs.

To construct an example we must first divide the set of positive numbers into two parts in biunique correspondence. Taking these parts, say, to be the alternate intervals $(n, n + 1]$ and for the correspondence, using an obvious linear mapping, we are led to a function whose graph is sketched in Figure 4. (The heavy dots on the ends of the segments indicate that these end points are included.)

The problem has continuous solutions if we work in the complex domain. On the other hand there exist analytic g such that (1) has a continuous solution in the real domain, but none at all in the complex domain. Such is $g(x) = x^2$. In the real domain take $f(x) = |x|^{2^{\frac{1}{2}}}$. In the complex domain no f exists as there is but one 2-cycle (namely, the complex cube roots of unity.)

Iterates of order $1/m$. It is not hard to generalize from (1) to

$$f^m(x) = g(x).$$

We state without proof the partial result:

Each orbit L_0 *with respect to* f *is the union of orbits* $L_1, \ldots, L_p$ *with respect to* g *and* p *is a divisor of* m. *If* $p < m$, L_0 *is cyclic. When* L_0 *is cyclic of order* n, $L_1, \ldots, L_p$ *are all cyclic of order* n/p, *and*

$$p = (m, n).$$

The oddness of n in Theorem 4 follows from the special instance of this last equation: $p = 1$, $m = 2$.

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