Involutive solutions of the Yang–Baxter equation of multipermutation level 2 and their permutation groups

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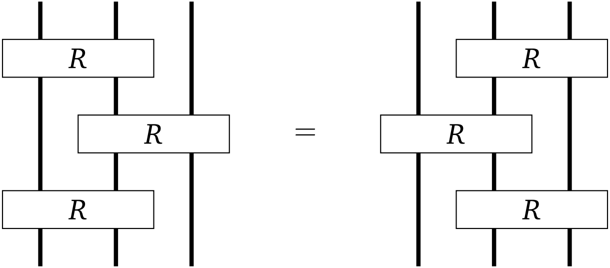
Yang–Baxter equation

Definition

Let V be a vector space. A homomorphism $R : V \otimes V \to V \otimes V$ is called a *solution of Yang–Baxter equation* if it satisfies

$$(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R).$$



Set-theoretic solutions

Definition

Let X be a set. A mapping $r : X \times X \to X \times X$ is called a *set-theoretic solution of Yang–Baxter equation* if it satisfies

$$(r \times \mathrm{id}_X)(\mathrm{id}_X \times r)(r \times \mathrm{id}_X) = (\mathrm{id}_X \times r)(r \times \mathrm{id}_X)(\mathrm{id}_X \times r).$$

A solution $r : (x,y) \mapsto (\sigma_x(y), \tau_y(x))$ is called *non-degenerate* if σ_x and τ_y are bijections, for all $x, y \in X$. A solution is called *involutive* if $r^2 = \mathrm{id}_{X^2}$.

Observation

If r is involutive then $\tau_y(x) = \sigma^{-1}_{\sigma_x(y)}(x)$.

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If r is involutive then $\tau_y(x) = \sigma^{-1}_{\sigma_x(y)}(x)$.

Retracts of involutive solutions

Definition

Let (X, σ, τ) be an involutive solution. We define a relation $\sim$ on X as

$$x \sim y \text{ if and only if } \sigma_x = \sigma_y.$$

The set $\{[x]_\sim \mid x \in X\}$ with operations

$$\sigma_{[x]_\sim}([y]_\sim) = [\sigma_x(y)]_\sim \qquad \text{and} \qquad \tau_{[y]_\sim}([x]_\sim) = [\tau_y(x)]_\sim$$

is called the *retract* of X and denoted by $\mathrm{Ret}(X)$.

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Retract is a solution

Theorem (Etingof, Schedler, Soloviev)

Let (X, σ, τ) be an involutive solution. Then $\mathrm{Ret}(X)$ *is a well-defined involutive solution.*

Definition

We say that an involutive solution (X, σ, τ) has *multipermutation level k* if k is the smallest integer such that $|\mathrm{Ret}^k(X)| = 1$.

Sketch of the proof.

$$\begin{array}{ccc} X & \to & \mathrm{Ret}(X) \\ \downarrow & & \downarrow \\ G(X) & \to & \mathcal{G}(X) \end{array}$$

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Retracts of non-involutive solutions

Definition

Let (X, σ, τ) be a solution. We define a relation $\sim$ on X as

$$x \sim y \text{ if and only if } \sigma_x = \sigma_y \text{ and } \tau_x = \tau_y$$

Theorem (P. J., A. P., A. Z.-D.)

Let (X, σ, τ) be a solution. Then $\mathrm{Ret}(X)$ is a well-defined solution.

Sketch of the proof.

Let $x \sim x'$ and $y \sim y'$. Then

- $\sigma_x(y) \sim \sigma_{x'}(y')$
- $\sigma_x^{-1}(y) \sim \sigma_{x'}^{-1}(y')$
- $\tau_y(x) \sim \tau_{y'}(x')$
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Multipermutation solutions of level 1

Proposition

Let X be a set and let f be a permutation on X. We define, for all $x, y \in X$,

$$\sigma_x(y) = f(y) \qquad \text{and} \qquad \tau_y(x) = f^{-1}(x).$$

Then (X, σ, τ) is an involutive solution of multipermutation level 1. Such a solution is called Lyubashenko *solution or* permutation *solution.*

On the other hand, every multipermutation solution of level 1 is a permutation solution.

Definition

If $f = \mathrm{id}_X$ then X is called a *trivial* solution.

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Reductivity

Definition

Let (X, σ, τ) be an involutive solution. We say that X is *k-reductive* if

$$\sigma_{\sigma_{\dots\sigma_{\sigma_{x_0}(x_1)}(x_2)\dots}(x_{k-1})}(x_k) = \sigma_{\sigma_{\dots\sigma_{x_1}(x_2)\dots}(x_{k-1})}(x_k)$$

Proposition (T. Gateva-Ivanova)

multipermutation level at most $k-1 \Rightarrow$ k-reductivity $\Rightarrow$ multipermutation level at most k

Proposition (T. Gateva-Ivanova)

Let (X, σ, τ) be an involutive solution satisfying

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Permutation group

Definition

Let (X, σ, τ) be an involutive solution. The group

$$\mathcal{G}(X) = \langle \sigma_x \mid x \in X \rangle$$

is called the *permutation group* of X or the *involutive Yang-Baxter group* of X.

Observation

Let (X, σ, τ) be a k-reductive involutive solution. Then each orbit of the action of $\mathcal{G}(X)$ is a subsolution of X of multipermutation level at most $k - 1$.

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2-reductive solutions

Proposition (T. Gateva-Ivanova)

Let (X, σ, τ) be an involutive solution. Then the following conditions are equivalent:

- *X is 2-reductive, i.e. $\sigma_{\sigma_x(y)}(z) = \sigma_y(z)$,*
- *$\sigma_x \in \mathrm{Aut}(X)$, for each $x \in X$, i.e. $\sigma_x\sigma_y(z) = \sigma_{\sigma_x(y)}\sigma_x(z)$,*
- *X has multip. level at most 2 and, for all $x \in X$, $\tau_x = \sigma_x^{-1}$,*
- *$\mathrm{Ret}(X)$ is a trivial solution.*

Corollary

Let (X, σ, τ) be a 2-reductive involutive solution. Then $\mathcal{G}(X)$ is abelian.

Theorem (W. Rump)

For each $k \in \mathbb{N}$, there exists an involutive solution of multipermutation level k with cyclic permutation group.

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Construction of 2-reductive solutions

Theorem (P. J., A. P., A. Z.-D.)

Let us have

- *an index set I,*
- *abelian groups A_i, for $i \in I$,*
- *a matrix of constants $c_{i,j} \in A_j$, for $i, j \in I$.*

Then the set $X = \bigsqcup_{i \in I} A_i$ with operation $\sigma : X \times X \to X$ defined by

$$\sigma_a(b) = b + c_{i,j}, \qquad \text{for } a \in A_i \text{ and } b \in A_j$$

is a 2-reductive involutive solution.

Conversely, every 2-reductive involutive solution can be obtained this way.

Corollary

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Numbers of 2-reductive solutions

n	1	2	3	4	5	6	7	8
involutive solutions	1	2	5	23	88	595	3456	34528
multip. level 2	1	2	5	19	70	359	2095	16332
2-reductive	1	2	5	17	65	323	1960	15421
mp level 2, not 2-red.	0	0	0	2	5	36	135	911

n	9	10	11
2-reductive	155889	2064688	35982357

n	12	13	14
2-reductive	832698007	25731050861	1067863092309

Theorem (S. Blackburn)

There are at least $2^{n^2/4+o(n\cdot\log n)}$ 2-reductive involutive solutions.

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Displacement group

Definition

Let (X,σ,τ) be an involutive solution. Then *displacement* group or the *transvection* group of X is the group

$$\mathrm{Dis}(X) = \langle \sigma_x \sigma_y^{-1} \mid x,y \in X \rangle.$$

Theorem (P. J., A. P.)

Let (X,σ,τ) be an involutive solution of multipermutation level at most 2. Then $\mathrm{Dis}(X)$ is a normal abelian subgroup of $\mathcal{G}(X)$. Moreover, $\mathcal{G}(X) = \mathrm{Dis}(X)\langle \sigma_x \rangle$, for any $x \in X$.

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Example on groups

Example

Let $X = \{1, 2, 3, 4, 5\}$ and let

$$\begin{aligned} \sigma_1 &= (1, 2)(3, 5) \\ \sigma_2 &= (1, 2)(4, 5) \\ \sigma_3 = \sigma_4 = \sigma_5 &= (1, 2)(3, 4) \end{aligned}$$

Then

$$\mathcal{G}(X) = \{\mathrm{id}_X, (1, 2)(3, 5), (1, 2)(4, 5), (1, 2)(3, 4), (3, 4, 5), (5, 4, 3)\}$$

and

$$\mathrm{Dis}(X) = \{\mathrm{id}_X, (3, 4, 5), (5, 4, 3)\}.$$

From multipermutation level 2 to 2-reductivity

Proposition (P. J., A. P. A. Z.-D.)

Let (X, σ, τ) be an involutive solution of multipermutation level at most 2 and choose $e \in X$. Let (X', σ', τ') be the following:

- $X' = X,$
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- $\tau'_y = \sigma_e \tau_{\sigma_e^{-1}(y)}.$

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Let (X, σ, τ) be a 2-reductive involutive solution and let $\pi \in S_X$ satisfy $\sigma_{\pi(y)} \pi \sigma_x = \sigma_{\pi(x)} \pi \sigma_y$. Let (X', σ', τ') be:

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Example on isotopy

Example

Let $X = \{1, 2, 3, 4, 5\}$ and let $\sigma_1 = (1, 2)(3, 5)$, $\sigma_2 = (1, 2)(4, 5)$,
$\sigma_3 = \sigma_4 = \sigma_5 = (1, 2)(3, 4)$.
Let $\sigma'_x = \sigma_x \sigma_1^{-1}$, then

$$\sigma'_1 = \mathrm{id}_{X'}$$
$$\sigma'_2 = (3, 4, 5)$$
$$\sigma'_3 = \sigma'_4 = \sigma'_5 = (5, 4, 3)$$

Let $\sigma''_x = \sigma_x \sigma_3^{-1}$, then

$$\sigma''_1 = (3, 4, 5)$$
$$\sigma''_2 = (5, 4, 3)$$
$$\sigma''_3 = \sigma''_4 = \sigma''_5 = \mathrm{id}_{X''}$$

Indecomposable solutions

Definition

We say that an ivolutive solution (X, σ, τ) is *indecomposable* if $\mathcal{G}(X)$ acts transitively on X.

Proposition

Let (X, σ, τ) be a k-reductive involutive solution of multipermutation level k. Then X is decomposable.

Proof.

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Theorem (M. Castelli, G. Pinto, W. Rump)

Let (X, σ, τ) be an indecomposable involutive solution of size pq, where p, q are primes, such that $\mathcal{G}(X)$ is abelian. Then X is of multipermutation level at most 2.

There is only one such solution, up to isomorphism if $p \neq q$, and there are $p + 1$ such solutions if $p = q$.

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Generators of the displacement group

Proposition (P. J., A. P.)

Let (X, σ, τ) be an indecomposable involutive solution of multipermutation level at most 2. Choose $e \in X$ and let $d = \sigma_e(e)$. Then $o(\sigma_e) = o(\sigma_d)$ and

$$\mathcal{G}(X) = \langle \sigma_e, \sigma_d \rangle \qquad \text{and} \qquad \mathrm{Dis}(X) = \left\langle \sigma_e^{-i} \sigma_d \sigma_e^{i-1} \mid i \in \mathbb{Z} \right\rangle .$$

Corollary

If $\mathcal{G}(X)$ is abelian then $\mathrm{Dis}(X)$ is cyclic and $\mathcal{G}(X) \cong C_1 \times C_2$, where C_1, C_2 are cyclic and $|C_1|$ divides $|C_2|$.

Observation

For finite solutions, there are 3 parameters:

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Let $n_1, n_2 \in \mathbb{Z}^+$ *be such that* $n_1 \mid n_2$*. Let* $r \in \{0, 1, \dots, n_2/n_1 - 1\}$ *be such that* $n_2 \mid n_1 r^2$*. Then* (X, σ, τ) *with* $X = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$ *and*

$$\sigma_{(a,i)}((b, j)) = (b - ar + i, j + ir - ar^2 + 1)$$

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Different choices of n_1*,* n_2 *and* r *give non-isomorphic solutions. Every finite indecomposable involutive solution of multipermutation level 2 with abelian permutation group is isomorphic to a solution so constructed.*

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Indecomposable solutions of size pq

Conditions: $|X| = n_1 \cdot n_2$, $n_1 \mid n_2$, $0 \leqslant r < \frac{n_2}{n_1}$, $n_2 \mid n_1 r^2$

Example

- **Case** $p \neq q$**:** $n_1 = 1$, $n_2 = pq$, $r = 0$
- **Case** $\mathbb{Z}_p \times \mathbb{Z}_p$**:** $n_1 = p$, $n_2 = p$, $r = 0$
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Theorem (P. J., A. P.)

There exists an indecomposable solution that homomorphically maps onto any indecomposable involutive solution of multipermutation level 2.

Idea of the proof.

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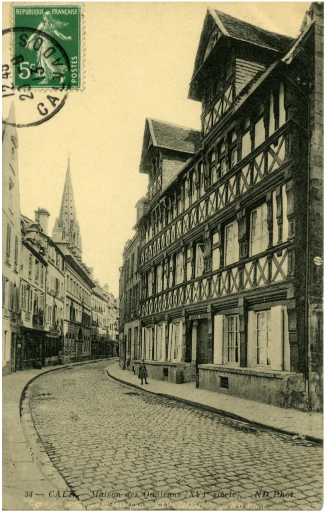
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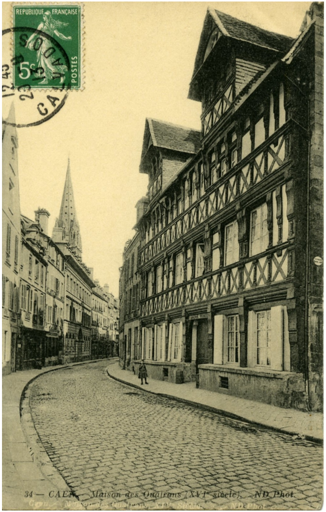
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