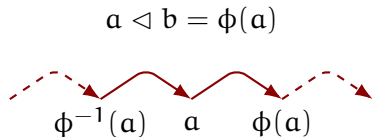
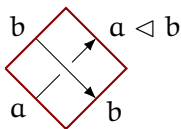


Unexpected applications of homotopical algebra to knot theory

(Honest title: Homology of permutation racks)

Victoria LEBED, University of Caen Normandy (France)
Joint work with Markus SZYMIK, NTNU (Norway)

(Virtual) Kyoto, February 2021



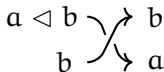
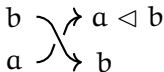
I. From topology to algebra

II. From algebra to topology

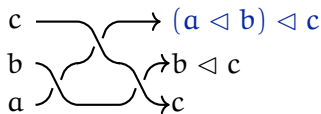
D. Joyce & S. Matveev, knot colorists separated by the Iron Curtain:

Take a set S endowed with a binary operation \triangleleft .

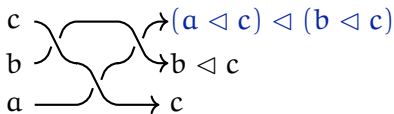
(S, \triangleleft) -colourings for
braid diagrams:



cf. Wirtinger
presentation
of $\pi_1([0, 1] \times \mathbb{R}^2 \setminus \beta)$:



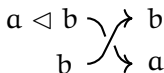
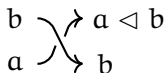
RIII
 \sim



D. Joyce & S. Matveev, knot colorists separated by the Iron Curtain:

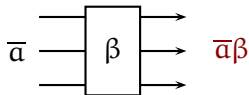
Take a set S endowed with a binary operation \triangleleft .

(S, \triangleleft) -colourings for
braid diagrams:



| | | | |
|------------------------------------|-------|---|--------------------------|
| $\text{End}(S^n) \leftarrow B_n^+$ | RIII | $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$ | shelf rack quandle |
| $\text{Aut}(S^n) \leftarrow B_n$ | & RII | $\forall b, a \mapsto a \triangleleft b$ is bijective | |
| $S \hookrightarrow (S^n)^{B_n}$ | & RI | $a \triangleleft a = a$ | |

$a \mapsto (a, \dots, a)$



| | | |
|------------------------------------|-------|---|
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shelf

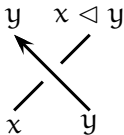
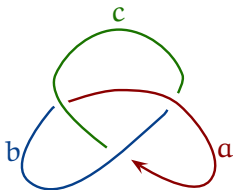
rack

quandle

$$a \mapsto (a, \dots, a)$$

| S | $a \triangleleft b$ | (S, \triangleleft) is a | in braid theory |
|------------------------------------|---------------------|---------------------------|--|
| $\mathbb{Z}[t^{\pm 1}] \text{Mod}$ | $ta + (1-t)b$ | quandle | Burau: $B_n \rightarrow GL_n(\mathbb{Z}[t^{\pm 1}])$ |
| group | $b^{-1}ab$ | quandle | Artin: $B_n \hookrightarrow \text{Aut}(F_n)$ |
| twisted linear quandle | | | Lawrence–Krammer–Bigelow |
| \mathbb{Z} | $a + 1$ | rack | $\text{lg}(w), \text{lk}_{i,j}$ |
| free shelf | | | Dehornoy: order on B_n |

(S, \triangleleft) -colourings for
knot diagrams:



$$a \triangleleft b = c, \quad b \triangleleft c = a, \quad c \triangleleft a = b$$

Proposition: (S, \triangleleft) is a quandle \implies
 $\# \{ (S, \triangleleft)\text{-colourings of diagrams} \}$ is a knot invariant.

Example: $(\mathbb{Z}_3, a \triangleleft b = 2b - a)$

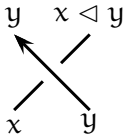
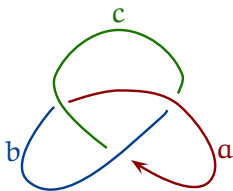


3 colourings



9 colourings

(S, \triangleleft) -colourings for
knot diagrams:

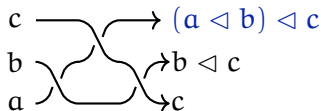


$$a \triangleleft b = c, \quad b \triangleleft c = a, \quad c \triangleleft a = b$$

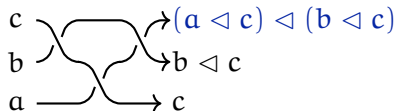
Theorem (Joyce & Matveev '82):

- $\# \text{Col}_{S, \triangleleft}(D) = \# \text{Hom}_{\text{Quandle}}(Q(K), S)$,
- $Q(K) =$ **fundamental quandle** of K
(a weak universal knot invariant).

The homology comes in



RIII
~



diagrams:

$D \xrightarrow{\text{R-move}} D'$

colorings:

$\mathcal{C} \rightsquigarrow \mathcal{C}'$

coloring sets:

$\text{Cols}_{S, \triangleleft}(D) \xleftrightarrow{1:1} \text{Cols}_{S, \triangleleft}(D')$

Counting invariants: $\# \text{Cols}_{S, \triangleleft}(D) = \# \text{Cols}_{S, \triangleleft}(D')$.

Question: Extract more information?

$$\omega(\mathcal{C}) = \omega(\mathcal{C}')$$

\Downarrow

$$\{ \omega(\mathcal{C}) \mid \mathcal{C} \in \text{Cols}_{S, \triangleleft}(D) \} = \{ \omega(\mathcal{C}') \mid \mathcal{C}' \in \text{Cols}_{S, \triangleleft}(D') \}.$$

Answer (Carter–Jelsovsky–Kamada–Langford–Saito '03): **State-sums** over crossings, and **Boltzmann weights**:

$$\phi: S \times S \rightarrow \mathbb{Z}_m \quad \rightsquigarrow \quad \omega_\phi(\mathcal{C}) = \sum_{\substack{b \\ a \times}} \pm \phi(a, b)$$

Conditions on ϕ :

$$\phi(a, b) + \phi(a \triangleleft b, c) + \cancel{\phi(b, c)} =$$

$$\cancel{\phi(b, c)} + \phi(a, c) + \phi(a \triangleleft c, b \triangleleft c)$$

$$\phi(a, a) =$$

$$0$$

Quandle cocycle invariants: $\{ \omega_\phi(\mathcal{C}) \mid \mathcal{C} \in \text{Col}_{S, \triangleleft}(D) \}$.

$$\phi: S \times S \rightarrow \mathbb{Z}_m \quad \rightsquigarrow \quad \omega_\phi(\mathcal{C}) = \sum_{\substack{b \\ a}} \pm \phi(a, b)$$

Quandle cocycle invariants: $\{ \omega_\phi(\mathcal{C}) \mid \mathcal{C} \in \text{Col}_{S, \triangleleft}(D) \}$.

Example: $\phi = 0 \rightsquigarrow$ counting invariants.

Quandle cocycle invariants \supsetneq counting invariants.

Conjecture (Clark-Saito-...):

Finite quandle cocycle invariants distinguish all knots.

Generalisation: $K^n \hookrightarrow \mathbb{R}^{n+2}$ and $\phi: S^{\times(n+1)} \rightarrow \mathbb{Z}_m$.

Wish:

$d^{n+1}\phi = 0 \implies \phi$ refines counting invariants for n -knots,

$\phi = d^n\psi \implies$ the refinement is trivial.

Fenn et al. '95 & *Carter et al.* '03 & *Graña* '00:

Shelf (S, \triangleleft) & abelian group $X \rightsquigarrow$ cochain complex

$$C_R^k(S, X) = \text{Map}(S^{\times k}, X),$$

$$\begin{aligned} (d_R^k f)(a_1, \dots, a_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i-1} (f(a_1, \dots, \hat{a}_i, \dots, a_{k+1}) \\ &\quad - f(a_1 \triangleleft a_i, \dots, a_{i-1} \triangleleft a_i, a_{i+1}, \dots, a_{k+1})) \end{aligned}$$

\rightsquigarrow Rack cohomology $H_R^k(S, X) = \text{Ker } d_R^k / \text{Im } d_R^{k-1}$.

Quandle (S, \triangleleft) & abelian group $X \rightsquigarrow$ sub-complex of (C_R^k, d_R^k) :

$$C_Q^k(S, X) = \{f: S^{\times k} \rightarrow X \mid f(\dots, a, a, \dots) = 0\}$$

\rightsquigarrow Quandle cohomology $H_Q^k(S, X)$.

This is what we were looking for! This construction yields:

- ✓ Boltzmann weights for constructing **higher knot invariants**
(powerfull and easy to compute);
- ✓ an important class of braided vector spaces giving nice **Hopf algebras**;
- ✓ a parametrization of abelian **rack extensions**.

Problem: Full rack/quandle (co)homology of a rack is hard to compute.

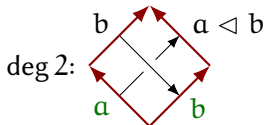
We will give a partial overview of available tools.

Fenn–Rourke–Sanderson '95:

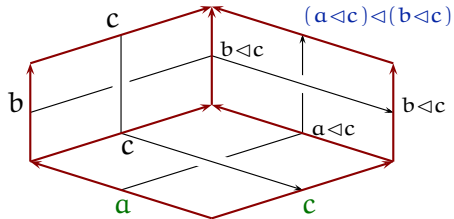
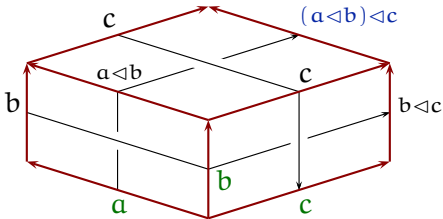
Shelf $(S, \triangleleft) \rightsquigarrow$ rack (= classifying) space $B(S)$. It is a CW-complex:

deg 0: *

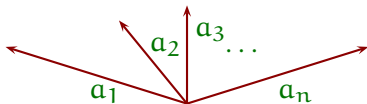
deg 1: $* \xrightarrow{a} *$



deg 3:



$$\text{deg } n: \coprod_{S \times n} [0, 1]^n$$



The coloring continues uniquely to other edges of $[0, 1]^n$.

Boundaries: usual topological ones.

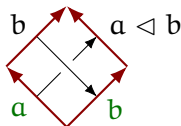
$$H_R^\bullet(S, X) \cong H^\bullet(B(S), X)$$

Nosaka '11: To get **quandle cohomology**, add 3-dimensional cells bounding



Proposition: $\pi_1(B(S)) \cong As(S)$,

where $As(S) := \langle S \mid a b = b (a \triangleleft b) \rangle$ is the associated group of (S, \triangleleft) .



Computations (*Fenn–Rourke–Sanderson '07*):

- 1) Trivial quandle $T_n = (\{1, \dots, n\}, a \triangleleft b = a)$: $B(T_n) \cong \Omega(\bigvee_n S^2)$.
- 2) Free rack on n generators FR_n : $B(FR_n) \cong \bigvee_n S^1$.

Theorem (Etingof–Graña '03): If (S, \triangleleft) is a rack and $\# \text{Inn}(S) \in X^*$, then

$$H_{\mathbb{R}}^k(S, X) \cong X^{r^k}$$

- ✓ $\text{Orb}(S) = \{ \text{orbits of } S \text{ w.r.t. } a \sim a \triangleleft b \}$, $r = \# \text{Orb}(S)$;
- ✓ $\text{Inn}(S)$ is the subgroup of $\text{Aut}(S)$ generated by $t_b: a \mapsto a \triangleleft b$.

Bad news: If $\# \text{Inn}(S) \in X^*$, then

quandle cocycle invariants = coloring invariants + linking numbers.

Hope: Look at $X = \mathbb{Z}_p$, or at the p -torsion of $H_{\mathbb{R}}^k(S, \mathbb{Z})$, where $p \mid \# \text{Inn}(S)$.

It works, and yields interesting invariants!

Theorem (*Szymik '19*): Quandle cohomology is a Quillen cohomology.

Applications:

- ✓ excision isomorphisms;
- ✓ Mayer–Vietoris exact sequences.

Homotopical tools: example

A permutation ϕ on a set $S \rightsquigarrow$ **permutation rack** $(S, \triangleleft_\phi \ b = \phi(a))$.

Theorem (*L.-Szymik '20*): $H_k^R((S, \triangleleft_\phi), X) \cong X^{\beta_k}$ where

$$\checkmark \beta_0 = 1, \beta_1 = r, \beta_{n+2} = (r-1)\beta_{n+1} + r_f\beta_n, \quad n \geq 0;$$

$$\checkmark r = \# \{ \text{orbits of } \phi \}, \quad r_f = \# \{ \text{finite orbits of } \phi \}.$$

Remark: $H_\bullet^R(S, \triangleleft_\phi)$ contains more information than $As(S, \triangleleft_\phi)$.

Sketch of proof:

Step 1 Explicit computations for **free permutation racks**
(= all orbits are infinite).

Trick: $H_k^R = \text{Ker } d_k^R / \text{Im } d_{k-1}^R$
study chains up to **boundaries**, then restrict to **cycles**
(usually: determine **cycles**, then mod out the **boundaries**).

Step 2 Choose a simplicial resolution by free permutations $F_\bullet \rightarrow S$

\leadsto a double complex $E_{p,q}^0 = C_q^R(F_p)$

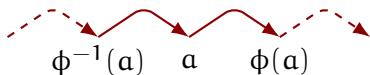
\leadsto two spectral sequences with the same target.

Step 3 Computations in the spectral sequences:

$$\text{1st SS: } E_{p,q}^\infty \cong \begin{cases} H_q^R(S) & \text{if } p = 0, \\ 0 & \text{if } p \neq 0. \end{cases}$$

$$\text{2nd SS: } E_{\bullet,q}^2 \cong \overline{H}_\bullet(S//\phi)^{\otimes(q-1)} \otimes H_\bullet(S//\phi),$$

where $S//\phi$ is the homotopy orbit space:



$$S//\phi = r_f \text{ circles } \sqcup r - r_f \text{ lines}$$

Step 4 For the 2nd SS, show that $E^\infty = E^2$.

For this, find enough independent elements in $H_q^R(S)$.