Lawrence representations of braid groups: self-distributive approach

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1 Historical overview

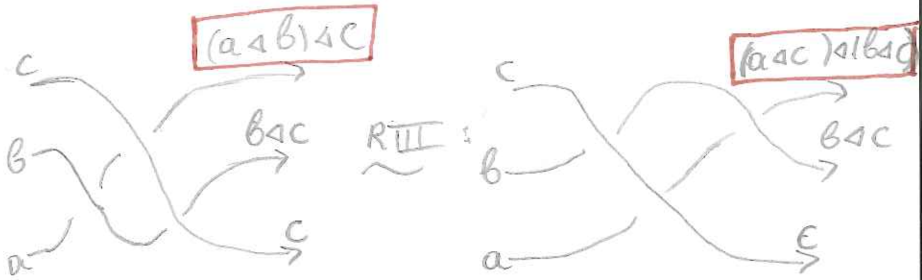
algebra	topology
• reps of Hecke algebras H_n	
'87 Jones (motivat^n: von Neumann alg.)	'90 Lawrence (PhD thesis)
• a generalisat^n for braid groups B_n ('94 Long-Moody)	
• certain Lawrence reps are faithful (and thus the B_n are linear !)	
'00 Krammer	'00 Bigelow
• yet another generalisat^n: a "representat^n enhancing machine" trivial rep. ⇝ ⇝ Lawrence rep.	
'08 Bigelow-Tian	?

Here: A combinatorial version of the Long-Moody-Bigelow-Tian machine (work in progress).

Secret goal: Advertise self-distributive structures.

2 Braids & self-distributivity

Diagram colourings by $(S, \triangleleft)$:
set ↗ ↖ Binary operat^n



cf. Wirtinger presentat^n of knot groups!

B_n-reps	R-moves	alg. axiom	alg. structure
$S^{\times n} \curvearrowleft B_n^+$	RIII	$(a\triangleleft b)\triangleleft c = (a\triangleleft c)\triangleleft(b\triangleleft c)$ (self-distributivity)	shelf
$S^{\times n} \curvearrowleft B_n$	RII	all right translations $x \mapsto x\triangleleft a$ are invertible	rack
$S \hookrightarrow (S^{\times n})^{B_n}$, $a \mapsto (a,\dots,a)$	RI	$a\triangleleft a = a$ (idempotence)	quandle

$S^{\times n} \ni \bar{a}$ $\boxed{B}$ $\bar{a}\cdot B \in S^{\times n}$, $B \in B_n^{(+)}$

Examples:

S	$a\triangleleft b$	name	in braid theory
group	$b^{-1}ab$	conjugat^n quandle	Artin rep.: $B_n \curvearrowright F_n$
$\mathbb{Z}[t^{\pm 1}]$-module	$ta+(1-t)b$	Alexander quandle	Burau rep.: $B_n \to GL_n(\mathbb{Z}[t^{\pm 1}])$
$\mathbb{Z}$	$a+1$	cyclic rack	$lg(w)$, $lk_{i,j}$ etc.
free shelf			Dehornoy order on B_n
$\{1,\dots,2^n\}$	$1\triangleleft b = b+1$	Laver table (shelf)	work in progress...
?	?	?	Lawrence-Krammer-Bigelow rep: $B_n \hookrightarrow GL_{\frac{n(n-1)}{2}}(\mathbb{Z}[q^{\pm}, t^{\pm}])$

To complete this table, we need to extend the notion of self-distributivity.

3. Twisted multi-distributivity

Fix a group G.

G-quandle = set S + binary operations $\underset{g}{\triangleleft}$ on S, $g \in G$, s.t.:

(1) $\boxed{(a \underset{g}{\triangleleft} b) \underset{h}{\triangleleft} c = (a \underset{h}{\triangleleft} c) \underset{h^{-1}gh}{\triangleleft} (b \underset{h}{\triangleleft} c)}$

(2) all right translations $x \mapsto x \underset{g}{\triangleleft} a$ are invertible

(3) $a \underset{g}{\triangleleft} a = a$.

Ex.: $S \in Mod_{\mathbb{Z}G} \rightsquigarrow$ Alexander G-quandle: $\boxed{a \underset{g}{\triangleleft} b = ag + b(1-g)}$

$\rightsquigarrow$ functor $Mod_{\mathbb{Z}G} \to G$-Quandles

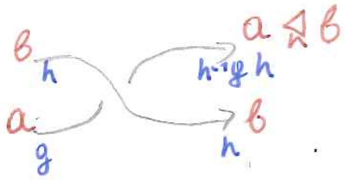
(with nice properties)

Rmk: • $\Rightarrow$ all $(S, \triangleleft_g)$ are quandles

• $\Leftrightarrow$ $(S \times G,\ (a,g) \triangleleft (b,h) = (a \underset{h}{\triangleleft} b,\ h^{-1}gh))$ is a quandle

• one can define S-quandles for any shelf S.

Double-layer colourings:



Lemma: They are compatible with Reidemeister moves.

Rmk: • Work for welded or virtual braids.

• Related to the holonomy Yang-Baxter equation (Kashaev-Reshetikhin, Turaev).

4 Digression: knotted graphs

Ishii et al., 2012: G-family of quandles = G-quandle $(S, \{\triangleleft_g\}_{g \in G})$ s.t.

$$\left.\begin{array}{l}(4)\ (a \triangleleft_g b) \triangleleft_h b = a \triangleleft_{gh} b \\ (5)\ a \triangleleft_1 b = a\end{array}\right\} \text{ i.e., } \begin{cases} G \xrightarrow[\text{monoid}]{} \mathrm{Bin}(S) \\ g \longmapsto \triangleleft_g \end{cases}$$

Ex.: Alexander G-quandle.

Motivation: knotted trivalent graphs.

Ex.:

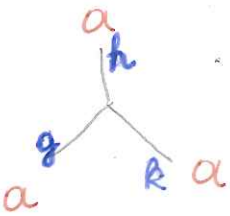
Θ_{st}

Θ_{KT} ← Kinoshita–Terasaka

Kauffman & Yamada & Yetter '89:

3-graphs $\overset{1:1}{\longleftrightarrow}$ diagrams $/_{RI-RVI}$

Double-layer colourings by a G-family of quandles:



$g^{\pm 1} h^{\pm 1} k^{\pm 1} = 1$

Lemma: They are compatible with Reidemeister moves.

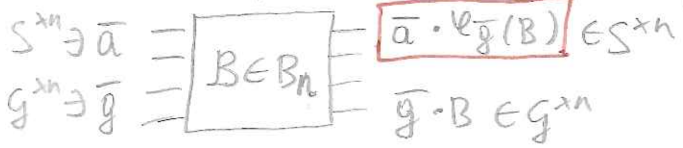
$\Rightarrow$ invariants of 3-graphs.

Even better: invariants of knotted handle-bodies

5 1-cocycles & representations of the B_n

Fix a G-quandle $(S, \{\triangleleft_g\}_{g\in G})$.

For any $\bar{g} \in G^{\times n}$, we have a map $\varphi_{\bar{g}} : B_n \to \mathrm{Aut}(S^{\times n})$:



<u>Lemma</u>: $\varphi_{\bar{g}}(BB') = \varphi_{\bar{g}}(B)\, \varphi_{\bar{g}\cdot B}(B')$ (1-cocycle property).

<u>Question</u>: Deduce an honest B_n-rep.?

An <u>answer</u> exist in a particular case:

- $G = F_n$, $\bar{g}^* = \underbrace{(x_1, \dots, x_n)}_{\text{generators of } F_n}$ $\Rightarrow \bar{g}^* \cdot B = (B(x_1), \dots, B(x_n))$
  Artin rep.
- S is an Alexander G-quandle $\Rightarrow \varphi_{\bar{g}} : B_n \to GL_n(\mathbb{Z}G)$.

<u>Thm</u>: In this situat^n, there is a group morphism

$$\varphi : B_n \to GL_n(\mathbb{Z}F_n) \rtimes B_n$$
$$B \mapsto (\varphi_{\bar{g}^*}(B), B).$$

$\square$ $(\varphi_{\bar{g}^*}(BB'), BB') \overset{\text{lemma}}{=} (\varphi_{\bar{g}^*}(B)\, \varphi_{\underbrace{\bar{g}^* \cdot B}_{(B(x_1), \dots, B(x_n))}}(B'), BB')$

$= (\varphi_{\bar{g}^*}(B)(B \cdot \varphi_{\bar{g}^*}(B')), BB') = (\varphi_{\bar{g}^*}(B), B)(\varphi_{\bar{g}^*}(B'), B')$. $\boxtimes$

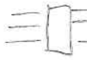
6) The LMBT machine

<u>Cor</u>: $\boxed{\rho : F_n \rtimes B_n \to \mathrm{Aut}(V) \overset{\varphi_*}{\rightsquigarrow} \rho^+ : B_n \to \mathrm{Aut}(V^{\oplus n})}$.

$\square$ $B_n \xrightarrow{\varphi} GL_n(\mathbb{Z}F_n) \rtimes B_n \hookrightarrow GL_n(\mathbb{Z}F_n \rtimes B_n) \xrightarrow{\rho} GL_n(\mathrm{Aut}\, V) \hookrightarrow \mathrm{Aut}(V^{\oplus n})$ $\boxtimes$

why care about this weird construction?

→ $F_n \rtimes B_n \hookrightarrow B_{n+1}$ in several ways $\Rightarrow$ one can start with $B_{n+1} \xrightarrow{\rho} \mathrm{Aut}(V)$.

e.g.  $\mapsto$ 

$x_i \mapsto$  i

→ ρ^+ is often richer than ρ:

- trivial B_{n+1}-rep. $\rightsquigarrow$ Burau B_n-rep.
- — — — P_{n+1}-rep. $\rightsquigarrow$ Gassner P_n-rep.
  +scaling ↑ pure braid group
- — — — B_2-rep. $\underbrace{\rightsquigarrow \rightsquigarrow}_{2 \text{ iterat}^{ns}}$ LKB B_n-rep.
  +shift + scaling

→ possibility of explicit computations:

$$\varphi(\sigma_i) = \left(\begin{pmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 0 & x_{i+1} & 0 \\ 0 & 1 & 1-x_{i+1} & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{pmatrix}, \sigma_i \right)$$

&
- $\sigma_i x_i = x_{i+1} \sigma_i$
- $\sigma_i x_{i+1} = x_{i+1}^{-1} x_i x_{i+1} \sigma_i$
- $\sigma_i x_j = x_j \sigma_i, \ j \notin \{i, i+1\}$

7. To be continued...

→ Extract topological information about the braid?
(Our combinatorial framework might be better adapted to it than Bigelow–Tian's algebraic one.)
Motivation: Krammer '02 & Ito–Wiest '12 :
classical & dual Garside length in terms of LKB matrices.

→ There is a "pseudo-Hecke" relat^n in $Mat_n(\mathbb{Z}F_n \rtimes B_n)$:

$$(\varphi(\sigma_i)+(g_{i+1},\sigma_i))(\varphi(\sigma_i)-(1,\sigma_i))=0.$$

Honest Hecke?

→ Other examples of G-quandles & associated B_n-reps?