TOWARDS BRAID-THEORETIC APPLICATIONS OF LAVER TABLES

LEBED Victoria

Joint work with DEHORNOY Patrick

Plan:

1. A Laver table is ...
2. Dreams: braid invariants based on LT.
3. Reality: 2- and 3- cocycles for LT.
4. Bonus: right-division ordering for LT.

---

1. **Shelf**: set S & operation $\triangleleft$ s.t. $\boxed{a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright (a \triangleright c)}$ (SD)

   Example: Group G, $f \triangleright g = fgf^{-1}$.

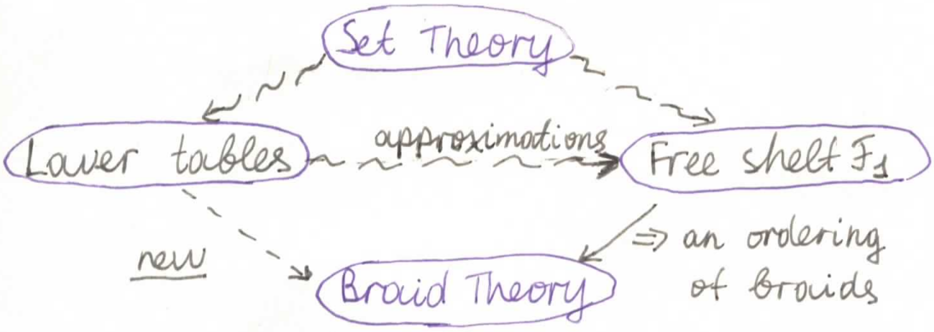
   $\mathcal{F}_1$: a free shelf generated by a single element γ.

   Laver table A_n ($n \in \mathbb{N}_0$): the unique shelf $(\{1,2,3,..,2^n\}, \triangleright)$ s.t.

   $\boxed{a \triangleright 1 \equiv a+1 \mod 2^n}$ (Init)

   Theorem (Laver, '95): (SD) & (Init) uniquely define $\triangleright$.

   ⚠ false for $\{1,2,3,..,q\}$, $q \neq 2^n$.



Richard LAVER

Patrick DEHORNOY

Why LT are natural: $\gamma = 1$ $\quad (\gamma \triangleright \gamma) \triangleright \gamma = 3$

$\gamma \triangleright \gamma = 2$ $\quad ((\gamma \triangleright \gamma) \triangleright \gamma) \triangleright \gamma = 4$ $\quad \ldots$

{Figure 1}: LT for $n \leq 4$.

= not using unprovable set-theoretic axiom I3

Elementary properties of LT:

- a projective system of shelves: $p_n : A_n \to A_{n-1}$, $a \mapsto a \bmod 2^{n-1}$
- periodic rows: $\boxed{\underbrace{p \triangleright 1}_{=p+1} < p \triangleright 2 < \cdots < \underbrace{p \triangleright 2^r}_{=2^n}}$ ... periodic repetition ...

  $\pi_n(p) := 2^r$ is the period of p in A_n.
- A_n is monogenerated: $\boxed{A_n \simeq \mathcal{F}_1 / \underbrace{(\ldots((\gamma \triangleright \gamma) \triangleright \gamma) \ldots) \triangleright \gamma}_{2^n+1} = \gamma}$

  $1 \longleftrightarrow \gamma$
- A. Drápal: $A_n \rightsquigarrow$ all finite monogenerated shelves.
- "nice" rows & columns:

A_n	1	2	$\ldots$	2^{n-1}	$\ldots$	2^n	
1	2			2^n		2^n	
2	3			2^n		2^n	
$\vdots$	$\vdots$			$\vdots$		$\vdots$	
2^{n-1}	$2^{n-1}+1$	$2^{n-1}+2$		2^n		2^n	$\pi_n(2^{n-1}) = 2^{n-1}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	
2^n-3	2^n-2	2^n		2^n		2^n	$\pi_n(2^n-3) = 2$
2^n-2	2^n-1	2^n		2^n		2^n	$\pi_n(2^n-2) = 2$
2^n-1	2^n	2^n		2^n		2^n	$\pi_n(2^n-1) = 1$
2^n	1	2	$\ldots$	2^{n-1}	$\ldots$	2^n	$\pi_n(2^n) = 2^n$

⚠ No closed formula for $p \triangleright q$, neither for $\pi_n(p)$.

Elementary conjectures:

- $\pi_n(1) \xrightarrow[n \to \infty]{} \infty$ Qu.: For which n one has $\pi_n(1) \geq 32$?
- $\pi_n(2) \geq \pi_n(1)$
- $\varprojlim_{n \in \mathbb{N}} A_n \supset \mathcal{F}_1$

⚠ Theorems under Axiom I3.

Conclusion: Rich combinatorics of LT.

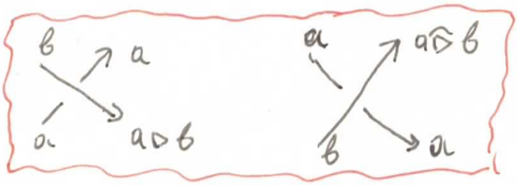
② Shelf colorings: 

Figure 2½: Shelf colorings for a RIII move.

RIII $\Leftrightarrow$ (SD)	rack
RII $\Leftrightarrow$ (Inv)	

$\boxed{a \widetilde{\triangleright} (a \triangleright b) = b = a \triangleright (a \widetilde{\triangleright} b)}$

rack (shelf) $\xrightarrow{\text{colorings}}$ (positive) braid invariants

Example: Group G, $f \triangleright g = f g f^{-1}$, $f \widetilde{\triangleright} g = f^{-1} g f$

colorings by $G \longleftrightarrow \mathrm{Rep}(\pi_1((\mathbb{R}^2 \times [0,1]) \setminus B), G)$

Wirtinger presentation

Problem: F_1 & A_n are shelves, but not racks.

Solution for F_1 (Dehornoy): partial colorings.

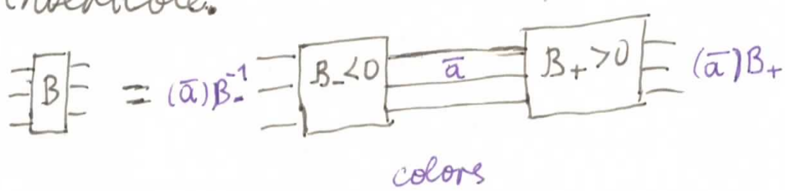
$(S, \triangleright)$ is a rack $\Leftrightarrow$ the color propagation map is bijective.

$$\sigma : S \times S \to S \times S,$$
$$(a,b) \mapsto (a \triangleright b, a).$$

For F_1, σ is injective $\Rightarrow$ partly invertible.

Normal form for braids:

$\rightsquigarrow$ colors $(\bar a) B_-^{-1}$ are B-propagable



Dehornoy's results:
- $\forall$ k-braids B, B', $\exists$ a common propagable $\bar a$
- $(\bar a)B = (\bar a)B' \Leftrightarrow B \simeq B'$.
- left division induces a total ordering on F_1 (& on $F_1^{\times k}$)

$\boxed{a \,|_e\, b \overset{\text{def}}{\Leftrightarrow} b = a \triangleright c}$

- relation $\boxed{B < B' \overset{\text{def}}{\Leftrightarrow} (\bar a)B \,|_e\, (\bar a)B'}$ is a total left-invariant ordering on B_n.

$B < B' \Rightarrow \alpha B < \alpha B'$.

What about A_n?

Problem: σ is not injective!

Motivation: • $A_n \xrightarrow[n \to \infty]{} A_\infty \supseteq F_1$ (conjecturally);

• A_n are finite.

③ Aim: add flexibility to coloring invariants.
Method: weights.

- 2-cocycles: $\varphi: S\times S \to \mathbb{Z}$ s.t. $\boxed{\varphi(a,c)+\varphi(a\triangleright b, a\triangleright c) = \varphi(b,c)+\varphi(a, b\triangleright c)}$
- 3-cocycles: $\psi: S\times S\times S \to \mathbb{Z}$ s.t. $\boxed{\begin{array}{l}\psi(a,b,c\triangleright d)+\psi(a,c,d)+\psi(a\triangleright b, a\triangleright c, a\triangleright d)\\ =\psi(a,b,d)+\psi(a,b\triangleright c, b\triangleright d)+\psi(b,c,d)\end{array}}$

Remark: part of rack cohomology theory (Fenn-Rourke-Sanderson, '95).

φ-weight (Carter-Jelsovsky-Kamada-Langford-Saito, '99):
ψ-weight

$$\begin{array}{c}S\text{-colored positive}\\ \text{braid diagram}\end{array} \longmapsto \sum_{\substack{b \;\;d\\ a}} \begin{array}{l}\varphi(a,b)\\ \psi(a,b,d)\end{array}$$

Figures 3&4: Invariance under RIII.

Shadow shelf colorings: $a \xrightarrow[a\triangleright d]{d}$

Shelf & 2- or 3-cocycle $\xrightarrow[\text{\& weights}]{\text{colorings}}$ pos. braid invariants.

Theorem (Dehornoy-L., '14):

2) $\boxed{Z^2_R(A_n)\simeq \mathbb{Z}^{2^n}}$ Basis: $\varphi_{const}(a,b)=1$ & coboundaries

$$\varphi_{q,n}(a,b)=\begin{cases}1 & \text{if } q\in Col(b),\ q\notin Col(a\triangleright b);\\ 0 & \text{otherwise.}\end{cases} \qquad 1\leq q<2^n$$

3) $\boxed{Z^3_R(A_n)\simeq \mathbb{Z}^{2^{2n}-2^n+1}}$ basis: ψ_{const} & explicit $\{0,\pm 1\}$-valued coboundaries

H) $\boxed{H^k_R(A_n)\simeq \mathbb{Z}}$, $k\leq 3$.

Theorem (L., '14):

k) $\boxed{Z^k_R(A_n)\simeq \mathbb{Z}^{P_k(2^n)}}$ $P_k(x)=\dfrac{x^k+x^{\lambda(k)}}{x+1}$, $\lambda(k)=\begin{cases}1 & \text{if } k \text{ is even;}\\ 0 & \text{otherwise.}\end{cases}$

H) $\boxed{H^k_R(A_n)\simeq \mathbb{Z}}$ for all k.

Remark: 2-cocycles capture the combinatorics of LT.

Example: $\pi_n(p) = \min\{q \mid \varphi_{2^{n-1};n}(p,q)=1\}$, $p<2^n$.

{Figure 5}: $\varphi_{9,3}$.

④ Right division: $\boxed{a \mid_r b \overset{def}{\iff} b = c \triangleright a}$

Theorem (Dehornoy – L., '14):

1) $\mid_r$ is a partial ordering for A_n.

2) $\boxed{a \mid_r b \iff Col(a) \supseteq Col(b).}$

3) $a \neq b \Rightarrow Col(a) \neq Col(b)$.

{Figure 6}: $\mid_r$ for $n \leq 4$.

Properties: • min. element: 1, max. element: 2^n.
• not linear for $n \geq 3$.
• not lattice for $n \geq 5$.

	$\mid_r$	$\mid_\ell$
A_n	is a partial ordering $\rightsquigarrow$ a good basis for Z^2_R	induces a trivial relation
F_1	induces a partial ordering $\rightsquigarrow$?	induces a total ordering $\rightsquigarrow$ an ordering of braids