TOWARDS TOPOLOGICAL APPLICATIONS OF LAVER TABLES

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Seminar,
GWU, Washington DC
21/01/2014

Plan:

1▷ Laver table is...

2▷ Dream: braids & knot invariants

3▷ Reality: 2- and 3- rack cocycles

1▷ Laver table is...

Thm (Laver, '95): $n \geq 0$, $A_n = \{1, 2, .., 2^n\}$

- $\exists!$ binary operation $\triangleright_n$ on A_n s.t.
  (SD) $a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright (a \triangleright c)$ $\rightsquigarrow$ shelf
  (Init) $a \triangleright 1 = a+1$
  ] $(A_n, \triangleright_n)$ is a Laver table
- line p: $\boxed{\overset{p \triangleright 1}{\overset{\shortparallel}{p+1}} < \dots < \overset{p \triangleright 2^k}{\overset{\shortparallel}{2^n}}}$ $\underbrace{\qquad\qquad}_{2^k}$ & periodicity] $\pi_n(p) = 2^k$ is the period of p in A_n

A_n		q	
		$\vdots$	
p	$\dots$	$p \triangleright_n q$	$\dots$
		$\vdots$	

A_0	1
1	1

A_1	1	2
1	2	2
2	1	2

$(p \triangleright_1 q = p \Rightarrow q)$

A_2	1	2	3	4
1	2	4	2	4
2	3	4	3	4
3	4	4	4	4
4	1	2	3	4

Properties:

easy:
- $a \triangleright 2^n = 2^n$, $2^n \triangleright a = a$ ⋮ $\pi_n(2^n) = 2^n$
- $(2^n - 1) \triangleright a = 2^n$ ⋮ $\pi_n(2^n - 1) = 1$
- $\pi_n(2^{n-1}) = 2^{n-1}$, $\pi_n(2^n - 2) = \pi_n(2^n - 3) = 2$

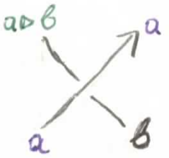
complex:
- $\boxed{\text{no f-la for } a \triangleright b}$
- $\boxed{\text{Conj: } \pi_n(1) \xrightarrow[n \to +\infty]{} +\infty \text{ \& } \pi_n(1) \geq \pi_n(2)}$ (Thm under a large cardinal axiom)
- $(A_n, \triangleright_n) \twoheadrightarrow (A_m, \triangleright_m)$ $n \geq m$
  $a \longmapsto a \bmod 2^m$
  $\rightsquigarrow A_\infty := \varprojlim (A_n, \triangleright_n)$
- $\boxed{\text{Conj.: } (111\dots) \text{ generates a free subshelf of } A_\infty}$ (Thm under ...)
- generated by 1

Thm (Drápal): $\boxed{\text{LT} \xrightarrow[\text{construction}]{\text{canonical}} \text{all monogenerated shelves}}$

$\rightarrow$ evidence of rich combinatorics

Origins: elementary embeddings:
$\{S \overset{f}{\hookrightarrow} S\}$, $f \triangleright g = \begin{cases} f \circ g \circ f^{-1} & \text{on } \operatorname{Im} f \\ \operatorname{Id} & \text{on } (\operatorname{Im} f)^c \end{cases}$ $\rightsquigarrow$ a shelf
S = limit rank V_λ

② Dreams: Braid & knot invariants



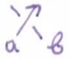
↧

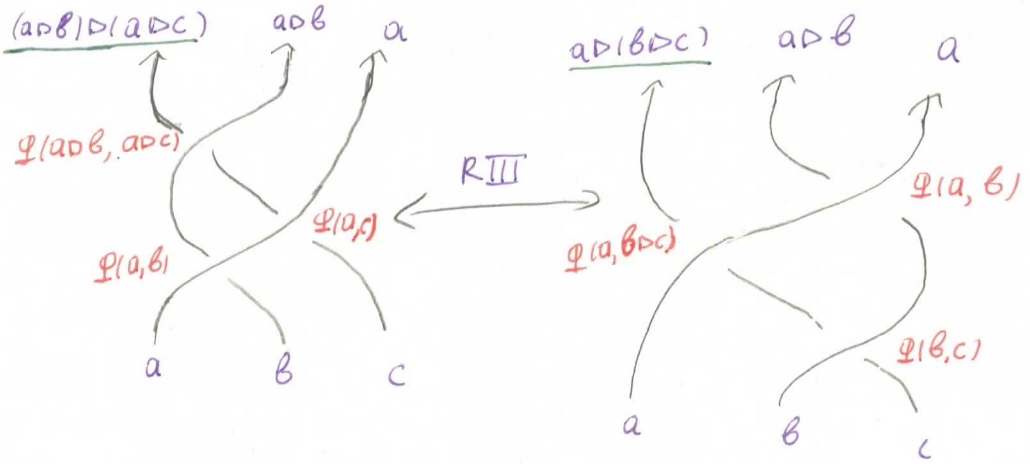
$\varphi(a,b)$

Braid / Knot

↧

$\sum \pm \varphi(a,b)$





RIII $\Longleftrightarrow$ (SD)

& $\underline{\varphi(a \triangleright b, a \triangleright c) + \varphi(a,c) = \varphi(b,c) + \varphi(a, b \triangleright c)}$ (*) } positive braids

RII $\Longleftrightarrow$ $(a,b) \overset{\sigma}{\mapsto} (a \triangleright b, a)$ is invertible

$\Longleftrightarrow$ $\forall\, a, (b \mapsto a \triangleright b$ is invertible$)$ } Braids

$(A_n, \triangleright_n)$: $b \mapsto a \triangleright_n b$ is neither injective, nor surjective

Can one still work with braids?

① Freeshelf$_1$: $b \mapsto a \triangleright b$ is inj. but not surj. $\Rightarrow$ σ^{-1} is partially defined + normal form of braids ⇝? "universal" ⇝ partial braid invariant ⇝? a total left-invariant order on braids] Dehornoy

② Using weights.

$\boxed{3}$ Reality: 2- and 3- rack cocycles

$Z_2(A_n) = \{\varphi : \mathbb{Z}(A_n \times A_n) \to \mathbb{Z} \text{ s.t. } (*)\}$ ← 2-cocycles of A_n

$\cup\!\!\!\mid$

$B_2(A_n) = \{\varphi(\alpha(x,y)) = \alpha(\theta(x \triangleright y) - \theta(y)) \mid \theta : A_n \to \mathbb{Z}\}$ ← 2- coboundaries of A_n

↳ $\text{weight}_\varphi(\text{knot}) = 0$

rack cochain complex for A_n.

<u>Thm 2</u>: ▸ $B^2(A_n)$ is freely generated over $\mathbb{Z}$ by $\Psi_q : (x,y) \mapsto \begin{cases} 1 \text{ if } q \in Col(y) \\ \quad \& \; q \notin Col(x \triangleright y) \\ 0 \text{ otherwise} \end{cases}$, $1 \leq q \leq 2^n - 1$ — rank $2^n - 1$

▸ $Z^2(A_n)$ — " — & const: $(x,y) \mapsto 1$. — 2^n

<u>Cor</u>: $H^2(A_n) \simeq \mathbb{Z}[\text{const}]$ <u>Rmk</u>: $H_2(A_n) \simeq \mathbb{Z}$

<u>Thm 3</u>: A similar description of $B^3(A_n)$ & $Z^3(A_n)$, rank $= 2^n(2^n - 1)$ $(+1)$.

<u>Pre-thm 4</u>: B^4 & Z^4, rank $= 2^{3n} - 2^{2n} + 2^n - 1$ $(+1)$.

<u>Thm n</u>: ?

<u>Applications</u>: imaginary: invariants of braids; ← ? ~ rich structure

real: — " — positive braids.

<u>Rmk</u>: Encode deep combinatorial properties of A_n. ⇐ rich structure

<u>Ex.</u>: $\pi(p) = \min\{y \mid \varphi_{2^n-1}(p,y) = 1\}$

Idea of proof (thm 2).

<u>Step 1</u>: $\varphi_q(x,y) = \delta_{y,q} - \delta_{x \triangleright y, q}$ <u>Rmk</u>: $\in \{0, \pm 1\}$.

→ coboundaries

→ φ_q, $1 \leq q \leq 2^n - 1$ & c_n are linearly indep. on row $2^n - 1$.

$\underset{\uparrow q}{0 \dots 0\,1\,0 \dots 0}$ $1\,1 \dots 1$

→ cocycle & zero on row $2^n - 1$ ⇒ zero

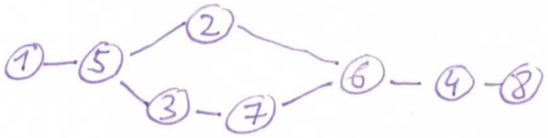
Step 2: $\varphi_q \rightsquigarrow \Psi_q$

Def.: $q \Vdash_n r \Leftrightarrow Col(q) \ni r \Leftrightarrow \exists p$ s.t. $p \triangleright q = r \Leftrightarrow q$ is a right divisor of r.

Thm: $\Vdash_n$ induces a partial order on A_n.

Applicⁿ: $\Psi_q = \sum_{r \Vdash q} \varphi_r$ is an alternative basis for $B^2(A_n)$.

Hasse diagram for A_3:



Rmk.: not a lattice for $n \geq 5$.

Rmk.: $p \vDash_n r \overset{def^n}{\Leftrightarrow} Row(p) \ni r \Leftrightarrow p$ is a left divisor of r

	$\Vdash$	$\vDash$
A_n	PO $\rightsquigarrow$ a nice basis of $Z^2(A_n)$ $\rightsquigarrow$???	not an order: $a \vDash 2^n \vDash b$, $a \nvDash a$ (for $a \neq 2^n$)
Free Shelf$_1$	induces a PO	TO $\rightsquigarrow$ Braid ordering (Dehornoy)

Idea of proof (thm).

→ transitivity

Prop.: $p \circ_n q := p \triangleright_n (q+1) - 1$ satisfies

$$\begin{cases} (1) & (p \circ_n q) \triangleright_n r = p \triangleright_n (q \triangleright_n r) \\ (2) & p \triangleright_n (q \circ_n r) = (p \triangleright_n q) \circ_n (p \triangleright_n r) \\ (3) & p \circ_n q = (p \triangleright_n q) \circ_n p \end{cases}$$

Let $p \Vdash q \Vdash r$. Then $r = s \triangleright q = s \triangleright (t \triangleright p) = (s \circ t) \triangleright p \Rightarrow p \Vdash r$.

→ anti-symmetry

Lemma: $p \Vdash q \nRightarrow Col(p) \supseteq Col(q)$.

Lemma: $Col(p) = Col(p + 2^{n-1}) \sqcup \{p\}$, $p \leq 2^{n-1}$.

$\Big\downarrow$ induction using $A_n \twoheadrightarrow A_{n-1}$

Prop.: $Col(p) \neq Col(q)$ for $p \neq q$.