Bialgebraic approach to rack cohomology

Victoria LEBED, Université Caen Normandie

Joint work with Simon Covez, Marco Farinati, and Dominique Manchon

$$(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$$

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1 Self-distributivity

Shelf: Set S with a binary operation $\triangleleft$ satisfying

Self-distributivity: $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$

Examples:

➻ group S with $x \triangleleft y = y^{-1}xy$ yields a quandle: (SD)
& $\forall y,\ x \mapsto x \triangleleft y$ is a bijection
& $x \triangleleft x = x$;

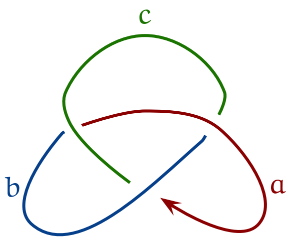
$$z^{-1}(y^{-1}xy)z = (z^{-1}y^{-1}z)(z^{-1}xz)(z^{-1}yz)$$

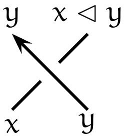
➻ abelian group S, $t\colon S \to S$, $\quad a \triangleleft b = ta + (1-t)b$.

Applications:

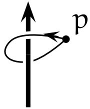
➻ invariants of knots and knotted surfaces (*Joyce & Matveev '82*);

$(S, \triangleleft)$-colourings of knot diagrams:

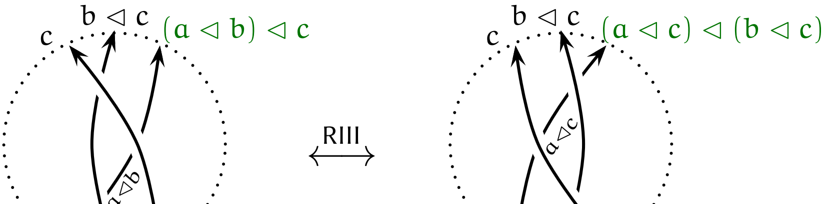




cf. Wirtinger presentation of $\pi_1(\mathbb{R}^3 \setminus K)$:



<u>Proposition</u>: $(S, \triangleleft)$ is a quandle $\implies$
$\#\{\, (S, \triangleleft)\text{-colourings of diagrams} \,\}$ is a knot invariant.



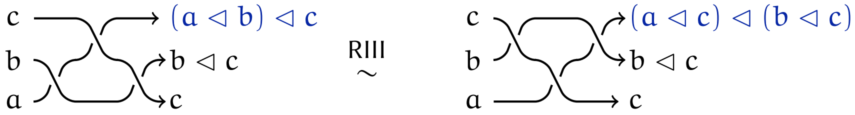
Self-distributivity: $(x \lhd y) \lhd z = (x \lhd z) \lhd (y \lhd z)$

Applications:

- invariants of knots and knotted surfaces (*Joyce & Matveev '82*);
- a total order on braid groups (*Dehornoy '91*);
- Hopf algebra classification (*Andruskiewitsch–Graña '03*);
- integration of Leibniz algebras (*Kinyon '07*);
- study of set-theoretic solutions to the Yang–Baxter equation.

You Could Have Invented SD Cohomology If You Were...

2 ... a Knot Theorist



diagrams: $D \overset{\text{R-move}}{\rightsquigarrow} D'$

colourings: $\mathcal{C} \rightsquigarrow \mathcal{C}'$

colouring sets: $\mathrm{Col}_{S,\triangleleft}(D) \overset{1:1}{\longleftrightarrow} \mathrm{Col}_{S,\triangleleft}(D')$

Counting invariants: $\#\,\mathrm{Col}_{S,\triangleleft}(D) = \#\,\mathrm{Col}_{S,\triangleleft}(D')$.

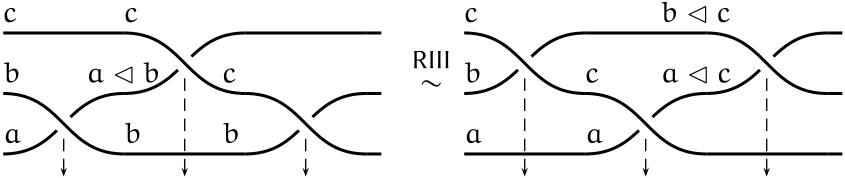
Question: Extract more information?

$$\omega(\mathcal{C}) = \omega(\mathcal{C}')$$
$$\Downarrow$$
$$\{\, \omega(\mathcal{C}) \,|\, \mathcal{C} \in \mathrm{Col}_{S,\triangleleft}(D) \,\} = \{\, \omega(\mathcal{C}') \,|\, \mathcal{C}' \in \mathrm{Col}_{S,\triangleleft}(D') \,\}.$$

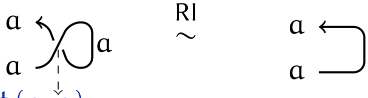
Answer (*Carter–Jelsovsky–Kamada–Langford–Saito ’03*): State-sums over crossings, and Boltzmann weights:

$$\phi\colon S \times S \to \mathbb{Z}_m \qquad \rightsquigarrow \qquad \omega_\phi(\mathcal{C}) = \sum \pm\phi(a,b)$$

Conditions on ϕ:



$$\phi(a,b)+\phi(a \triangleleft b,c)+\cancel{\phi(b,c)} = \cancel{\phi(b,c)}+\phi(a,c)+\phi(a \triangleleft c, b \triangleleft c)$$



$$\phi(a,a) = 0$$

Quandle cocycle invariants: $\left\{\, \omega_\phi(\mathcal{C}) \;\middle|\; \mathcal{C} \in \mathrm{Col}_{S,\triangleleft}(D) \,\right\}$.

$$\phi\colon S \times S \to \mathbb{Z}_m \qquad \rightsquigarrow \qquad \omega_\phi(\mathcal{C}) = \sum \pm\phi(a, b)$$

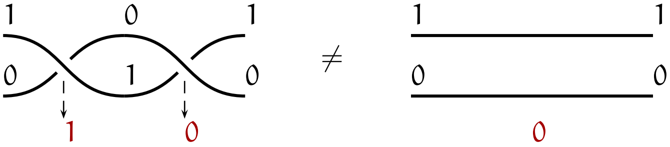
Quandle cocycle invariants: $\left\{ \omega_\phi(\mathcal{C}) \,\middle|\, \mathcal{C} \in \mathrm{Col}_{S,\triangleleft}(D) \right\}$.

Example: $\phi = 0 \qquad \rightsquigarrow \qquad$ counting invariants.

Quandle cocycle invariants $\supsetneq$ counting invariants.

Example: $S = \{0, 1\}$, $a \triangleleft b = a$,

$\phi(0, 1) = 1$ and $\phi(a, b) = 0$ elsewhere.



$$\phi\colon S \times S \to \mathbb{Z}_m \qquad \rightsquigarrow \qquad \omega_\phi(\mathcal{C}) = \sum \pm\phi(a, b)$$

Quandle cocycle invariants: $\{\, \omega_\phi(\mathcal{C}) \mid \mathcal{C} \in \mathrm{Col}_{S,\triangleleft}(D) \,\}$.

Conjecture (*Clark–Saito–…*):

Finite quandle cocycle invariants distinguish all knots.

Generalisation: $K^n \hookrightarrow \mathbb{R}^{n+2}$ and $\phi\colon S^{\times(n+1)} \to \mathbb{Z}_m$.

Wish:

$d^{n+1}\phi = 0 \implies$ ϕ refines counting invariants for n-knots,

$\phi = d^n\psi \implies$ the refinement is trivial.

3 ... a Hopf Algebraist

Question: Classify nice (= f.-d. pointed) Hopf algebras over $\mathbb{C}$.

Classification program (*Andruskiewitsch–Graña–Schneider ’98*):

nice Hopf algebra A

$\rightsquigarrow$

Yetter–Drinfel′d module $V \in {}^{H}_{H}\mathbf{YD}$

$\rightsquigarrow$

YBE solution (V, σ) $\quad\leftsquigarrow$ quandle $(S, \triangleleft)$ & $\phi \colon S \times S \to \mathbb{Z}_m$

$\rightsquigarrow$

Nichols algebra $B(V)$

$\rightsquigarrow$ $\quad$ & $V \in {}^{H}_{H}\mathbf{YD}$

bosonisation Hopf algebra $B(V)\#H$

✓ $G(A) =$ the group of group-like elements of A, $\quad H(A) = \mathbb{C}G(A)$;

✓ $R(A) =$ coinvariants of $\mathrm{gr}(A) \twoheadrightarrow \mathrm{gr}(A)_0 = H(A)$, $\quad V(A) = \mathrm{Prim}(R(A))$;

✓ in red: “arrows with a large image”;

✓ $\mathrm{gr}(A) \cong R(A)\#H(A) =$ [conjecturally] $= B(V(A))\#H(A)$.

YBE solution $(\mathbb{C}S, \sigma_{\lhd,\phi})$ $\leftsquigarrow$ quandle $(S, \lhd)$ & $\phi\colon S \times S \to \mathbb{Z}_m$

$$\boxed{\sigma_{\lhd,\phi}\colon (a, b) \mapsto q^{\phi(a,b)}(b, a \lhd b)}$$

Here q is an mth root of unity, or transcendental.

Wish:

$$d^2\phi = 0 \implies (\mathbb{C}S, \sigma_{\lhd,\phi}) \text{ is a YBE solution,}$$
$$\phi - \phi' = d^1\psi \implies \text{the YBE solutions are isomorphic.}$$

4 The desired cohomology theory

Fenn et al. '95 & Carter et al. '03 & Graña '00:

Shelf $(S, \lhd)$ & abelian group A $\rightsquigarrow$ cochain complex

$$C_R^k(S, A) = \mathrm{Map}(S^{\times k}, A),$$

$$(d_R^k f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} (f(a_1, \ldots, \widehat{a_i}, \ldots, a_{k+1})$$

$$- f(a_1 \lhd a_i, \ldots, a_{i-1} \lhd a_i, a_{i+1}, \ldots, a_{k+1}))$$

$\rightsquigarrow$ Rack cohomology $H_R^k(S, A) = \mathrm{Ker}\, d_R^k / \mathrm{Im}\, d_R^{k-1}$.

Quandle $(S, \lhd)$ & abelian group A $\rightsquigarrow$ sub-complex of (C_R^k, d_R^k):

$$C_Q^k(S, A) = \{ f \colon S^{\times k} \to A \mid f(\ldots, a, a, \ldots) = 0 \}$$

$\rightsquigarrow$ Quandle cohomology $H_Q^k(S, A)$.

$$C^k_R(S,A) = \mathrm{Map}(S^{\times k}, A),$$

$$(d^k_R f)(a_1,\ldots,a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} \big(f(a_1,\ldots,\widehat{a_i},\ldots,a_{k+1})$$
$$- f(a_1 \lhd a_i,\ldots,a_{i-1} \lhd a_i, a_{i+1},\ldots,a_{k+1})\big);$$

$$C^k_Q(S,A) = \{ f\colon S^{\times k} \to A \mid f(\ldots,a,a,\ldots) = 0 \}.$$

In small degree:

$(d^0_R f)(a_1) = 0$

$(d^1_R f)(a_1,a_2) = f(a_1 \lhd a_2) - f(a_1)$ $\quad H^1_R(S,A) \cong \mathrm{Map}(\mathrm{Orb}(S),A)$

$(d^2_R f)(\overline{a}) = f(a_1 \lhd a_2, a_3) - f(a_1,a_3) + f(a_1,a_2) - f(a_1 \lhd a_3, a_2 \lhd a_3)$

$f \in C^2_Q \iff f(a,a) = 0.$

Remark: $d^2_R d^1_R = 0 \iff$ self-distributivity for $\lhd$.

This is what we were looking for! This construction yields:

- ✓ Boltzmann weights for constructing higher knot invariants;
- ✓ an important class of YBE solutions giving nice Hopf algebras;
- ✓ also, a parametrisation of abelian shelf extensions.

5 Topological realisation

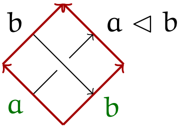
Fenn–Rourke–Sanderson '95:

Shelf $(S, \triangleleft)$ $\rightsquigarrow$ classifying space $B(S)$. It is a CW-complex:

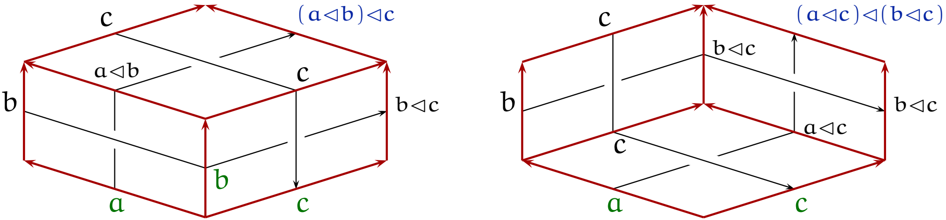
deg 0: $*$

deg 1: $* \xrightarrow{a} *$

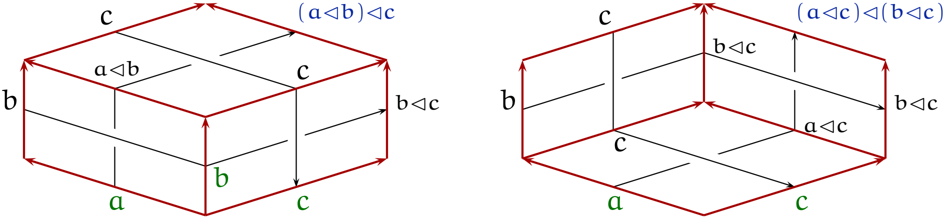
deg 2:



deg 3:

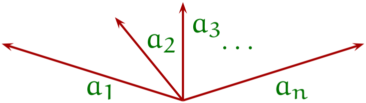


deg 3:



Remark: the edges can be coloured starting from the green corner $\iff$ $\lhd$ is self-distributive.

deg n: $\coprod_{S^{\times n}} [0,1]^n$

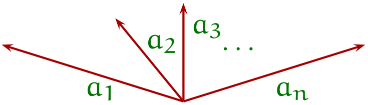


The colouring continues uniquely to other edges of $[0,1]^n$.

Boundaries: usual topological ones.

$$\boxed{H^*_R(S,A) \cong H^*(B(S),A)}$$

deg n: $\coprod_{S^{\times n}} [0,1]^n$



The colouring continues uniquely to other edges of $[0,1]^n$.

Boundaries: usual topological ones.

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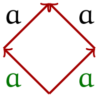
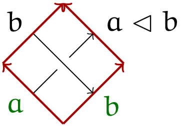
Nosaka '11: To get quandle cohomology, add 3-dimensional cells bounding



$$\boxed{H^*_R(S, A) \cong H^*(B(S), A)}$$

This brings topological tools in the study of H^*_R.

✓ $\boxed{\pi_1(B(S)) \cong As(S)}$ where $As(S) := \langle S \mid a\,b \;=\; b\,(a \lhd b)\rangle$ is the associated (= adjoint = structure = universal enveloping) group of $(S, \lhd)$.



✓ Rack cohomology becomes a pre-cubical cohomology, i.e.,

$$d^k_R = \sum_{i=1}^{k+1} (-1)^{i-1}(d^k_{i,0} - d^k_{i,1}), \qquad d_{i,\varepsilon} d_{j,\zeta} = d_{j-1,\zeta} d_{i,\varepsilon} \quad \text{for all } i < j.$$

✓ Concrete computations (*Fenn–Rourke–Sanderson '07*):

1) Trivial quandle $T_n = (\{1, \ldots, n\},\ a \lhd b = a)$: $\quad B(T_n) \cong \Omega(\vee_n \mathbb{S}^2)$.

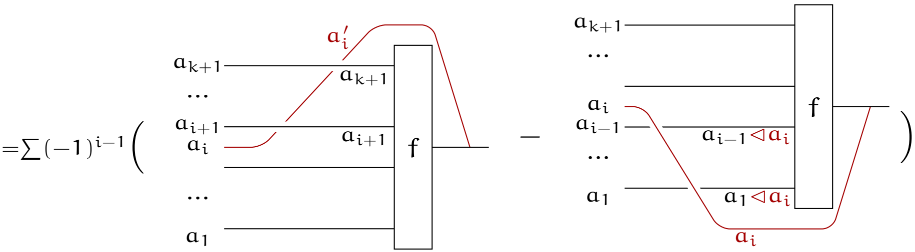
2) Free rack on n generators FR_n: $\quad B(FR_n) \cong \vee_n \mathbb{S}^1$.

6 Adding coefficients

Level 1: For $M \in {}_{\mathrm{As}(S)}\mathrm{Mod}_{\mathrm{As}(S)}$, the cohomology $H^*_R(S, M)$ is defined by

$$C^k_R(S, M) = \mathrm{Map}(S^{\times k}, M),$$

$$(d^k_R f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} \big(f(a_1, \ldots, \widehat{a_i}, \ldots, a_{k+1}) \cdot a_i' - a_i \cdot f(a_1 \triangleleft a_i, \ldots, a_{i-1} \triangleleft a_i, a_{i+1}, \ldots, a_{k+1}) \big)$$



$$a_i' = (\ldots(a_i \triangleleft a_{i+1})\ldots) \triangleleft a_{k+1}.$$

$$M \in {}_{As(S)}\mathrm{Mod}_{As(S)} \quad \rightsquigarrow \quad H^*_R(S, M).$$

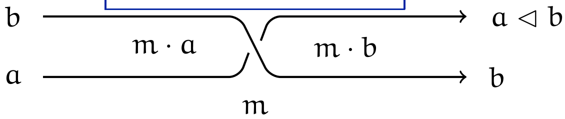
Many of the above constructions and results generalise to this setting, e.g. the classifying space:

deg 0: $m \in M$,

deg 1: $m \xrightarrow{a} m \cdot a$.

Application: arc-and-region colourings (*Carter–Kamada–Saito '01*).

$$\boxed{m \cdot ab = m \cdot b(a \lhd b)}$$



Examples of As(S)-(bi)modules:

✓ trivial actions;

✓ $As(S) \in {}_{As(S)}\mathrm{Mod}_{As(S)}$;

✓ $S \in \mathrm{Mod}_{As(S)}$, with the action induced by $a \cdot b = a \lhd b$;

✓ $\mathrm{Mod}_{\mathbb{C}[t^{\pm 1}]} \subset \mathrm{Mod}_{As(S)}$, with the action induced by $a \cdot b = ta$.

6 Adding coefficients

Level 2: M is a Beck module over S, i.e., an abelian group objects in the category **Rack** $\downarrow S$ (*Andruskiewitsch–Graña '03*, *Jackson '05*):

- ✓ $\rightsquigarrow$ $H^*_R(S, M)$;
- ✓ classification of a larger class of shelf extensions.

Pursuing the homotopical approach further:

Theorem (*Szymik '17*): Quandle cohomology is a Quillen cohomology.

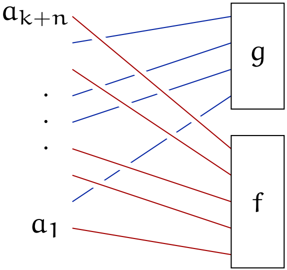
Applications:

- ✓ excision isomorphisms;
- ✓ Mayer–Vietoris exact sequences.

7 Cup product

$$\smile : C^k_R \otimes C^n_R \to C^{k+n}_R$$

$$f \smile g(a_1, \ldots, a_{k+n}) = \sum_{\text{splittings}} (-1)^{\#\times}$$



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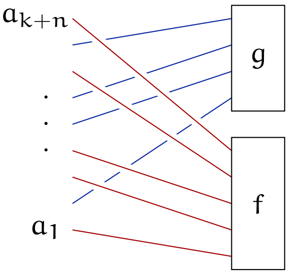

Theorem:

- ✓ $(C^*_R, \smile)$ d.g. associative, graded commutative up to an explicit htpy;
- ✓ $(H^*_R, \smile)$ associative, graded commutative;
- ✓ even better: C^*_R dendriform, Zinbiel up to an explicit htpy.

Interpretations:

- ✓ quantum shuffle coproduct;
- ✓ topological cup product;
- ✓ cup product in cubical cohomology;
- ✓ shelf $\rightsquigarrow$ explicit d.g. bialgebra $\rightsquigarrow$ cohomology + structure.

(*Serre '51*, *Baues '98*, *Clauwens '11*, *Covez '12*, *L. '17*,
Covez–Farinati–L.–Manchon '19.)

8 Our bialgebra

Shelf $(X, \lhd) \rightsquigarrow$ explicit d.g. bialgebra $B(X)$:

$$B(X) := \mathbb{Z}\langle x, e_y \; : \; x, y \in X\rangle \,/\, \langle y x^y - xy, \, y e_{x^y} - e_x y \; : \; x, y \in X\rangle.$$

Here $x^y = x \lhd y$.

Structure:

$$\begin{aligned} |e_x| &= 1, & |x| &= 0; \\ d(e_x) &= 1 - x, & d(x) &= 0; \\ \Delta(e_x) &= e_x \otimes x + 1 \otimes e_x, & \Delta(x) &= x \otimes x; \\ \varepsilon(e_x) &= 0, & \varepsilon(x) &= 1. \end{aligned}$$

$B(X)$ is a d.g. $As(X)$-bimodule:

$$x \cdot b \cdot y = xby \qquad \text{for all } x, y \in X, \; b \in B.$$

Shelf $(X, \lhd)$ $\rightsquigarrow$ explicit d.g. bialgebra and d.g. $As(X)$-bimodule $B(X)$:

$$B(X) := \mathbb{Z}\langle x, e_y \ : \ x, y \in X\rangle \,/\, \langle y x^y - xy, y e_{x^y} - e_x y \ : \ x, y \in X\rangle.$$

Proposition:

✓ $B(X)$ computes the rack homology with coeffs in the monoid

$$As^+(X) := \langle X \,|\, a\, b \ = \ b\,(a \lhd b)\rangle^+ :$$

$$H_\bullet(B(X)) \cong H^R_\bullet(X, As^+(X));$$

✓ the d.g. coalgebra $\overline{B}(X) := \mathbb{Z} \otimes_{As(X)} B(X)$ computes the rack homology with trivial coeffs:

$$H_\bullet(\overline{B}(X)) \cong H^R_\bullet(X, \mathbb{Z}).$$

$\rightsquigarrow$ the cup product $\smile$ on rack cohomology.

9 Explicit homotopy

Define $h\colon B \to B \otimes B$ (or $\overline{B} \to \overline{B} \otimes \overline{B}$) by

$$h(ae_{x_1} \cdots e_{x_n}) := \sum_{i=1}^{n} (-1)^i (a \otimes a)(\tau\Delta)(\cdots e_{x_{i-1}})(e_{x_i} \otimes e_{x_i})\Delta(e_{x_{i+1}} \cdots),$$

$$h(a) := 0,$$

where $a = y_1 \cdots y_m$.

Theorem: h is a homotopy between Δ and $\tau\Delta$:

$$(d \otimes \mathrm{Id}_B + \mathrm{Id}_B \otimes d)h + hd = \tau\Delta - \Delta.$$

Corollary: h measures the commutativity defect of $\smile$.

10 More structure and more explicit homotopies

$B(X)$ is a codendriform coalgebra and $\overline{B}(X)$ is a d.g. codendriform coalgebra:

$$\overleftarrow{\Delta}(a e_{x_1} \cdots e_{x_n}) = (a e_{x_1} \otimes a x_1)\Delta(e_{x_2} \cdots e_{x_n}),$$

$$\overrightarrow{\Delta}(a e_{x_1} \cdots e_{x_n}) = (a \otimes a e_{x_1})\Delta(e_{x_2} \cdots e_{x_n}),$$

Define $\overline{h}\colon \overline{B} \to \overline{B} \otimes \overline{B}$ by

$$\overline{h}(a e_{x_1} \cdots e_{x_n}) = -(a x_1 \otimes a e_{x_1}) h(e_{x_2} \cdots e_{x_n}).$$

Theorem: $\overline{h}$ is a homotopy between $\overrightarrow{\Delta}$ and $\tau\overleftarrow{\Delta}$:

$$(d \otimes \mathrm{Id}_{\overline{B}} + \mathrm{Id}_{\overline{B}} \otimes d)\overline{h} + \overline{h} d = \overrightarrow{\Delta} - \tau\overleftarrow{\Delta}.$$

Corollary: $\overline{h}$ measures the Zinbielity defect of $\smile$.

Remark: The codendriform structure is not surprising (cf. cubical or quantum shuffle interpretation), while the Zinbiel structure is!

11 Quandle cohomology vs rack cohomology

$$C^k_R(S,A) = \mathrm{Map}(S^{\times k}, A),$$
$$C^k_Q(S,A) = \{\, f\colon S^{\times k} \to A \mid f(\ldots, a, a, \ldots) = 0 \,\}$$

Theorem (*Litherland–Nelson '03*): The rack cohomology of a quandle splits:

$$\boxed{H^k_R \cong H^k_Q \oplus H^k_D}$$

Here H^k_D is the cohomology of an explicit degenerate subcomplex of C^k_R.

Generalisation (*L.–Vendramin '17*): A similar splitting holds for skew cubical cohomology, hence for a wide class of YBE solutions.

Theorem (*Przytycki–Putyra '16*): Degenerate cohomology is degenerate. That is, H^k_Q completely determines H^k_D.

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Theorem (*Przytycki–Putyra '16*): Degenerate cohomology is degenerate. That is, H^k_Q completely determines H^k_D.

Our theorem:

✓ H_Q is an associative subalgebra, and H_D is an associative ideal;

✓ H_Q is not a Zinbiel subalgebra, but H_D is a Zinbiel ideal.

Question: Does H_Q determine H_D as a Zinbiel algebra?