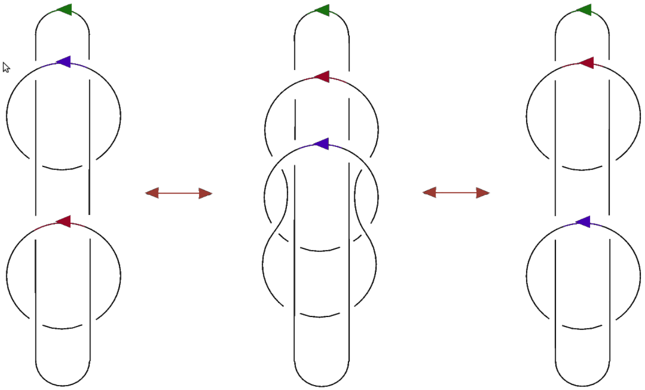
# On set-theoretic solutions to the Yang–Baxter equation

Victoria LEBED (Nantes, France & Dublin, Ireland)
with Leandro VENDRAMIN (Buenos Aires)



XXICLA, Buenos Aires, July 2016

# 1 Set-theoretic Yang–Baxter equation

✓ set $S$,

✓ $\sigma\colon S^{\times 2} \to S^{\times 2}$

Yang-Baxter equation (YBE)

$$\sigma_1 \circ \sigma_2 \circ \sigma_1 = \sigma_2 \circ \sigma_1 \circ \sigma_2 \colon S^{\times 3} \to S^{\times 3}$$

where $\sigma_1 = \sigma \times \mathrm{Id}_S$, $\sigma_2 = \mathrm{Id}_S \times \sigma$.

Origins: *Drinfel′d 1990*.

# 1 Set-theoretic Yang–Baxter equation

✓ set $S$,
✓ $\sigma\colon S^{\times 2} \to S^{\times 2}$

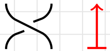
Yang-Baxter equation (YBE)

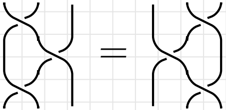
$$\sigma_1 \circ \sigma_2 \circ \sigma_1 = \sigma_2 \circ \sigma_1 \circ \sigma_2 \colon S^{\times 3} \to S^{\times 3}$$

where $\sigma_1 = \sigma \times \mathrm{Id}_S$, $\sigma_2 = \mathrm{Id}_S \times \sigma$.

Origins: *Drinfel′d 1990*.

$(S, \sigma)$: braided set.

$\sigma \longleftrightarrow$ 



(Reidemeister III)

# 1 Set-theoretic Yang–Baxter equation

✓ set $S$,

✓ $\sigma \colon S^{\times 2} \to S^{\times 2}$ $\qquad (x,y) \mapsto ({}^{x}y, x^{y})$

Yang-Baxter equation (YBE)

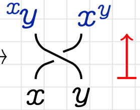
$$\sigma_1 \circ \sigma_2 \circ \sigma_1 = \sigma_2 \circ \sigma_1 \circ \sigma_2 \colon S^{\times 3} \to S^{\times 3}$$

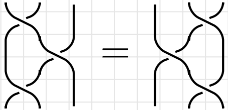
where $\sigma_1 = \sigma \times \mathrm{Id}_S$, $\sigma_2 = \mathrm{Id}_S \times \sigma$.

Origins: *Drinfel′d 1990*.

$(S, \sigma)$: braided set.

$\sigma \longleftrightarrow$



(Reidemeister III)

# 1 Set-theoretic Yang–Baxter equation

✓ set $S$,

✓ $\sigma\colon S^{\times 2} \to S^{\times 2}$ $\qquad (x,y) \mapsto ({}^{x}y, x^{y})$

Yang-Baxter equation (YBE)

$$\sigma_1 \circ \sigma_2 \circ \sigma_1 = \sigma_2 \circ \sigma_1 \circ \sigma_2 \colon S^{\times 3} \to S^{\times 3}$$

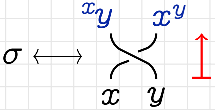
where $\sigma_1 = \sigma \times \mathrm{Id}_S$, $\sigma_2 = \mathrm{Id}_S \times \sigma$.

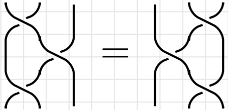
Origins: *Drinfel′d 1990*.

$(S, \sigma)$: braided set.

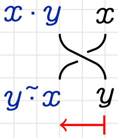
➜ Left non-degenerate:

$x \mapsto x^y$ bijection for all $y$.




(Reidemeister III)



# 1 Set-theoretic Yang–Baxter equation

✓ set $S$,

✓ $\sigma\colon S^{\times 2} \to S^{\times 2}$  $(x, y) \mapsto ({}^{x}y, x^{y})$

Yang-Baxter equation (YBE)

$$\sigma_1 \circ \sigma_2 \circ \sigma_1 = \sigma_2 \circ \sigma_1 \circ \sigma_2 \colon S^{\times 3} \to S^{\times 3}$$

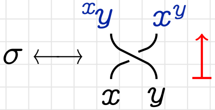
where $\sigma_1 = \sigma \times \mathrm{Id}_S$, $\sigma_2 = \mathrm{Id}_S \times \sigma$.

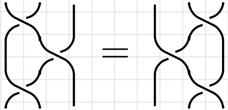
Origins: *Drinfel′d 1990*.

$(S, \sigma)$: braided set.

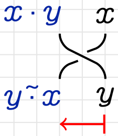
➙ Left non-degenerate:
$x \mapsto x^{y}$ bijection for all $y$.

➙ Birack: $\sigma$ invertible and
left & right non-degenerate.




(Reidemeister III)



# 2 Structure (semi)group

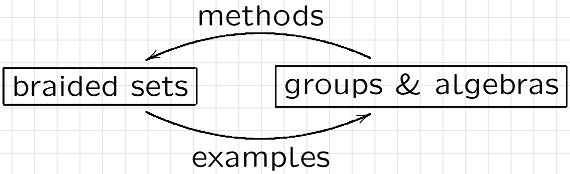
Structure (semi)group of $(S,\sigma)$: $(S)G_{S,\sigma} = \langle\, S \mid xy = {}^{x}y x^{y} \,\rangle$

methods

braided sets ← groups & algebras

# 2 Structure (semi)group

Structure (semi)group of $(S,\sigma)$: $(S)G_{S,\sigma} = \langle\, S \mid xy = {}^{x}y x^{y} \,\rangle$




Theorem: $(S,\sigma)$ a finite RI-compatible birack, $\sigma^2 = \mathrm{Id} \implies$

- ✓ $SG_{S,\sigma}$ is of $I$-type, cancellative, Öre;
- ✓ $G_{S,\sigma}$ is solvable, Garside;
- ✓ $\Bbbk SG_{S,\sigma}$ is Koszul, noetherian, Cohen–Macaulay, Artin–Schelter regular

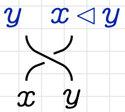
(*Manin, Gateva-Ivanova & Van den Bergh, Etingof–Schedler–Soloviev, Jespers–Okniński, Chouraqui 80'-...*).

## ③ Self-distributive structures

**Shelf**: set $S$ & $S \times S \xrightarrow{\triangleleft} S$ s.t.

$$(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$$

$\Leftrightarrow$ $\sigma_{\triangleleft} =$



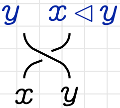

is a braiding on $S$

# 3 Self-distributive structures

Shelf: set $S$ & $S \times S \xrightarrow{\triangleleft} S$ s.t.

$$(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$$

Rack: & $\forall y,\ x \mapsto x \triangleleft y$ bijective.

$\Leftrightarrow \quad \sigma_{\triangleleft} =$ 


is a braiding on $S$

$\Leftrightarrow$ LND $\Leftrightarrow$ birack

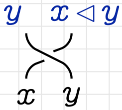
# 3 Self-distributive structures

Shelf: set $S$ & $S\times S \stackrel{\triangleleft}{\to} S$ s.t.

$$(x\triangleleft y)\triangleleft z = (x\triangleleft z)\triangleleft(y\triangleleft z)$$

Rack: & $\forall y,\ x\mapsto x\triangleleft y$ bijective.

Quandle: & $x\triangleleft x = x$.

$\Leftrightarrow \quad \sigma_{\triangleleft} =$ 


is a braiding on $S$

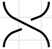
$\Leftrightarrow$ LND $\Leftrightarrow$ birack

$\Leftrightarrow (x,x) \stackrel{\sigma_{\triangleleft}}{\mapsto} (x,x)$.

## 3 Self-distributive structures

Shelf: set $S$ & $S \times S \overset{\lhd}{\to} S$ s.t.

$$(x \lhd y) \lhd z = (x \lhd z) \lhd (y \lhd z)$$

$\Leftrightarrow$ $\sigma_{\lhd} =$ 


is a braiding on $S$

Rack: & $\forall y$, $x \mapsto x \lhd y$ bijective.

Quandle: & $x \lhd x = x$.

$\Leftrightarrow$ LND $\Leftrightarrow$ birack

$\Leftrightarrow$ $(x, x) \overset{\sigma_{\lhd}}{\mapsto} (x, x)$.

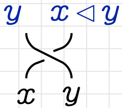
Examples:

- $\mathbb{Z}[t]$-module $S$ & $a \lhd b = ta + (1 - t)b$;

# ③ Self-distributive structures

Shelf: set $S$ & $S \times S \xrightarrow{\lhd} S$ s.t.

$$(x \lhd y) \lhd z = (x \lhd z) \lhd (y \lhd z)$$

$\Leftrightarrow$ $\sigma_{\lhd} =$ 


is a braiding on $S$

Rack: & $\forall y,\ x \mapsto x \lhd y$ bijective.

$\Leftrightarrow$ LND $\Leftrightarrow$ birack

Quandle: & $x \lhd x = x$.

$\Leftrightarrow$ $(x, x) \overset{\sigma_{\lhd}}{\mapsto} (x, x)$.

Examples:

- $\mathbb{Z}[t]$-module $S$ & $a \lhd b = ta + (1-t)b$;
- group $S$ & $x \lhd y = y^{-1}xy$:

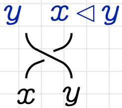
# 3 Self-distributive structures

Shelf: set $S$ & $S \times S \overset{\triangleleft}{\to} S$ s.t.

$$(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$$

Rack: & $\forall y,\ x \mapsto x \triangleleft y$ bijective.

Quandle: & $x \triangleleft x = x$.

$\Leftrightarrow \quad \sigma_{\triangleleft} =$ 


is a braiding on $S$

$\Leftrightarrow$ LND $\Leftrightarrow$ birack

$\Leftrightarrow (x, x) \overset{\sigma_{\triangleleft}}{\mapsto} (x, x)$.

Examples:

- $\mathbb{Z}[t]$-module $S$ & $a \triangleleft b = ta + (1 - t)b$;
- group $S$ & $x \triangleleft y = y^{-1}xy$:

  rack = "(w)rack and ruin of a group"

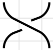
# 3 Self-distributive structures

Shelf: set $S$ & $S \times S \xrightarrow{\lhd} S$ s.t.

$$(x \lhd y) \lhd z = (x \lhd z) \lhd (y \lhd z)$$

Rack: & $\forall y$, $x \mapsto x \lhd y$ bijective.

Quandle: & $x \lhd x = x$.

$\Leftrightarrow$ $\sigma_{\lhd} =$ 


is a braiding on $S$

$\Leftrightarrow$ LND $\Leftrightarrow$ birack

$\Leftrightarrow$ $(x,x) \overset{\sigma_{\lhd}}{\mapsto} (x,x)$.

Examples:

- $\mathbb{Z}[t]$-module $S$ & $a \lhd b = ta + (1-t)b$;
- group $S$ & $x \lhd y = y^{-1}xy$:

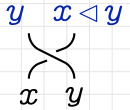
  rack = "(w)rack and ruin of a group"

  adjunction **Groups** $\rightleftarrows$ **Quandles**

# 3 Self-distributive structures

Shelf: set $S$ & $S \times S \overset{\lhd}{\to} S$ s.t.

$$(x \lhd y) \lhd z = (x \lhd z) \lhd (y \lhd z)$$

$\Leftrightarrow$ $\sigma_\lhd =$ 


is a braiding on $S$

Rack: & $\forall y$, $x \mapsto x \lhd y$ bijective. $\Leftrightarrow$ LND $\Leftrightarrow$ birack

Quandle: & $x \lhd x = x$. $\Leftrightarrow$ $(x,x) \overset{\sigma_\lhd}{\mapsto} (x,x)$.

Examples:

- $\mathbb{Z}[t]$-module $S$ & $a \lhd b = ta + (1-t)b$;
- group $S$ & $x \lhd y = y^{-1}xy$:

  rack = "(w)rack and ruin of a group"

  adjunction **Groups** $\rightleftarrows$ **Quandles**

  $G_{S,\sigma_\lhd} = \langle\, S \mid x \lhd y = y^{-1}xy \,\rangle$

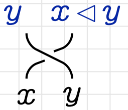
# 3 Self-distributive structures

Shelf: set $S$ & $S \times S \overset{\triangleleft}{\to} S$ s.t.

$$(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$$

Rack: & $\forall y,\ x \mapsto x \triangleleft y$ bijective.

Quandle: & $x \triangleleft x = x$.

$\Leftrightarrow \quad \sigma_{\triangleleft} =$ 


is a braiding on $S$

$\Leftrightarrow$ LND $\Leftrightarrow$ birack

$\Leftrightarrow (x,x) \overset{\sigma_{\triangleleft}}{\mapsto} (x,x)$.

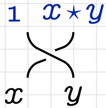
Applications:

- invariants of knots and knotted surfaces (*Joyce & Matveev 1982*);
- study of large cardinals (*Laver 1980s*);
- Hopf algebra classification (*Andruskiewitsch–Graña 2003*).

# 4 Monoids

For a monoid $(S, \star, 1)$,
the associativity of $\star$

$\Leftrightarrow$ $\sigma_\star =$



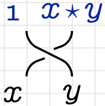

is an involutive braiding on $S$

# 4 Monoids

For a monoid $(S, \star, 1)$,
the associativity of $\star$

$S$ is a group

$\Leftrightarrow \quad \sigma_\star =$

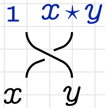



is an involutive braiding on $S$

$\Rightarrow$ LND

# 4 Monoids

For a monoid $(S, \star, 1)$, the associativity of $\star$ $\Leftrightarrow$ $\sigma_\star =$




is an involutive braiding on $S$

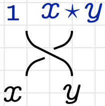
$S$ is a group $\Rightarrow$ LND

$$SG_{S,\sigma_\star} \xrightarrow{\sim} S,$$

$$S^{\times k} \ni x_1 \cdots x_k \mapsto x_1 \star \cdots \star x_k.$$

# 4 Monoids

For a monoid $(S, \star, 1)$,
the associativity of $\star$ $\Leftrightarrow$ $\sigma_\star =$




is an involutive braiding on $S$

$S$ is a group $\Rightarrow$ LND

$$SG_{S,\sigma_\star} \overset{\sim}{\to} S,$$

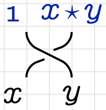
$$S^{\times k} \ni x_1 \cdots x_k \mapsto x_1 \star \cdots \star x_k.$$

Generalization: monoid factorization $G = HK$,
$S = H \cup K$, $\sigma(x, y) = (h, k)$, $h \in H$, $k \in K$, $hk = xy$;

$$S \simeq SG_{S,\sigma} / _{1_S = 1_{SG}}.$$

# 4 Monoids

For a monoid $(S, \star, 1)$, the associativity of $\star$ $\Leftrightarrow$ $\sigma_\star =$




is an involutive braiding on $S$

$S$ is a group $\Rightarrow$ LND

$$SG_{S,\sigma_\star} \xrightarrow{\sim} S,$$
$$S^{\times k} \ni x_1 \cdots x_k \mapsto x_1 \star \cdots \star x_k.$$

Generalization: monoid factorization $G = HK$,

$S = H \cup K$, $\sigma(x, y) = (h, k)$, $h \in H$, $k \in K$, $hk = xy$;

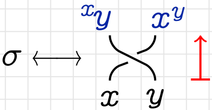
$$S \simeq SG_{S,\sigma} / _{1_S = 1_{SG}}.$$

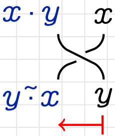
Other examples:

- ✓ cycle sets, braces;
- ✓ Young tableaux;
- ✓ distributive lattices.

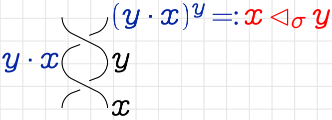
# 5 Associated shelf

Fix an LND braided set $(S, \sigma)$.



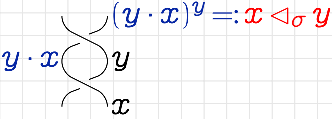

**Proposition (*L.-V. 2015*):** one has a shelf $(S, \triangleleft_\sigma)$, where



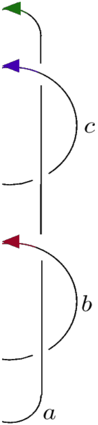
# 5 Associated shelf

Fix an LND braided set $(S, \sigma)$.

**Proposition (*L.-V. 2015*):** one has a shelf $(S, \lhd_\sigma)$, where



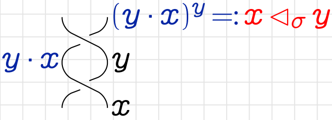

Proof:



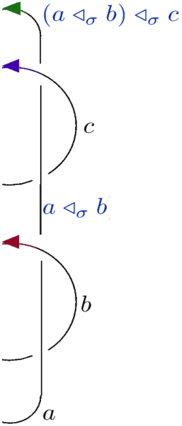
## 5 Associated shelf

Fix an LND braided set $(S, \sigma)$.

**Proposition (*L.-V. 2015*):** one has a shelf $(S, \lhd_\sigma)$, where



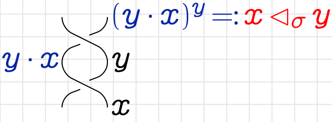

Proof:



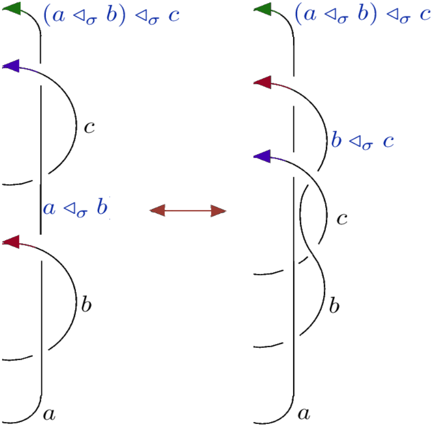
# 5 Associated shelf

Fix an LND braided set $(S, \sigma)$.

**Proposition (*L.-V. 2015*):** one has a shelf $(S, \lhd_\sigma)$, where




Proof:

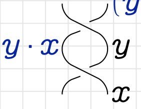


## 5 Associated shelf

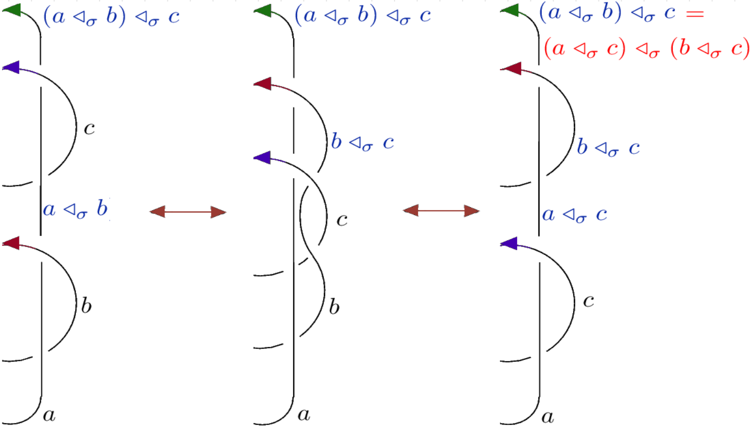
Fix an LND braided set $(S,\sigma)$.

**Proposition (*L.-V. 2015*):** one has a shelf $(S,\triangleleft_\sigma)$, where

$$(y\cdot x)^y =: x \triangleleft_\sigma y$$



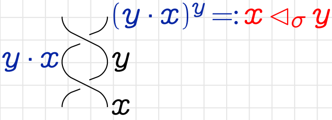

<u>Proof</u>:



## 5 Associated shelf

Fix an LND braided set $(S, \sigma)$.

**Proposition (*L.-V. 2015*):** one has a shelf $(S, \triangleleft_\sigma)$, where

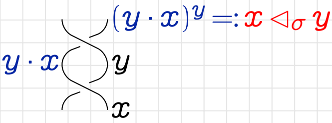



$\triangleleft_\sigma$ is a “shadow” of $\sigma$

# 5 Associated shelf

Fix an LND braided set $(S,\sigma)$.

**Proposition (*L.-V. 2015*):** one has a shelf $(S,\triangleleft_\sigma)$, where




$\boxed{\triangleleft_\sigma \text{ is a ``shadow'' of } \sigma}$

**Proposition (*L.-V. 2015*):**

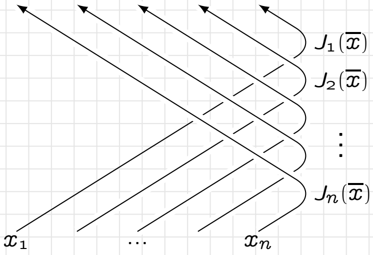
- $(S,\triangleleft_\sigma)$ is a rack $\Leftrightarrow$ $\sigma$ is invertible;
- $(S,\triangleleft_\sigma)$ is a trivial ($x \triangleleft_\sigma y = x$) $\Leftrightarrow$ $\sigma^2 = \mathrm{Id}$;
- $x \triangleleft_\sigma x = x$ $\Leftrightarrow$ $\sigma(x \cdot x, x) = (x \cdot x, x)$.

# 6 Guitar map

$$J^{(n)}\colon S^{\times n} \xrightarrow{\sim} S^{\times n},$$

$$(x_1,\ldots,x_n) \mapsto (x_1^{x_2\cdots x_n},\ldots,x_{n-1}^{x_n},x_n),$$

$$\text{where } x_i^{x_{i+1}\cdots x_n} = (\ldots(x_i^{x_{i+1}})\ldots)^{x_n}.$$

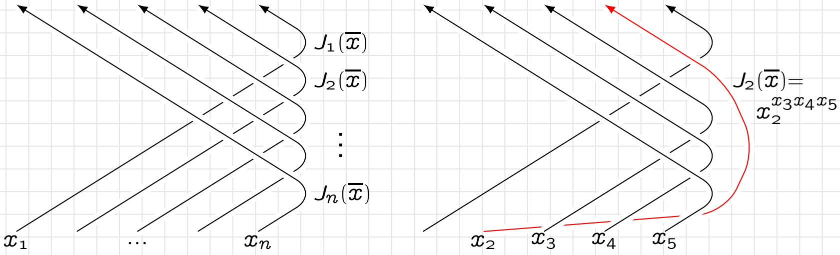


# 6 Guitar map

$$J^{(n)} : S^{\times n} \xrightarrow{\sim} S^{\times n},$$

$$(x_1, \ldots, x_n) \mapsto (x_1^{x_2 \cdots x_n}, \ldots, x_{n-1}^{x_n}, x_n),$$

$$\text{where } x_i^{x_{i+1} \cdots x_n} = (\ldots (x_i^{x_{i+1}}) \ldots)^{x_n}.$$

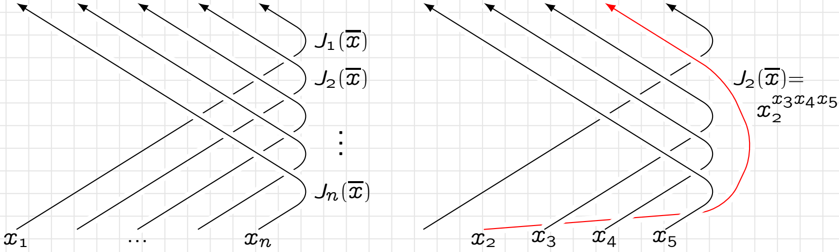


# 6 Guitar map

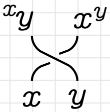
$$J^{(n)}: S^{\times n} \xrightarrow{\sim} S^{\times n},$$

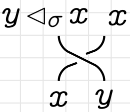
$$(x_1, \ldots, x_n) \mapsto (x_1^{x_2 \cdots x_n}, \ldots, x_{n-1}^{x_n}, x_n),$$

where $x_i^{x_{i+1} \cdots x_n} = (\ldots (x_i^{x_{i+1}}) \ldots)^{x_n}$.




**Proposition (*L.-V. 2015*):** $J\sigma_i = \sigma_i' J$.

$\sigma =$ 

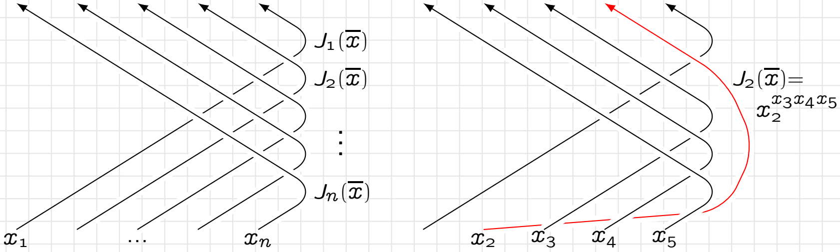

$\sigma' =$ 

## 6 Guitar map

$$J^{(n)}\colon S^{\times n} \xrightarrow{\sim} S^{\times n},$$

$$(x_1,\ldots,x_n) \mapsto (x_1^{x_2\cdots x_n},\ldots,x_{n-1}^{x_n},x_n),$$

where $x_i^{x_{i+1}\cdots x_n} = (\ldots(x_i^{x_{i+1}})\ldots)^{x_n}$.



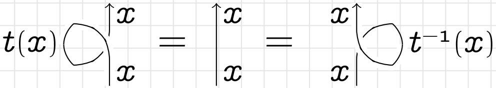

**Proposition (*L.-V. 2015*):** $\boxed{J\sigma_i = \sigma_i' J}$.

Corollary: $\sigma$ and $\sigma'$ yield isomorphic $B_n$-actions on $S^{\times n}$.

Warning: In general, $(S,\sigma) \ncong (S,\sigma')$ as braided sets!

# 7 RI-compatibility

RI-compatible braiding: $\exists t\colon S \xrightarrow{\sim} S$ s.t. $\sigma(t(x),x) = (t(x),x)$.

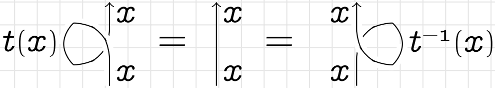



(Reidemeister I)

7

# RI-compatibility

RI-compatible braiding: $\exists t \colon S \xrightarrow{\sim} S$ s.t. $\sigma(t(x), x) = (t(x), x)$.




(Reidemeister I)

Example:

for a rack, it means $x \triangleleft x = x$ (here $t(x) = x$).

## 8 Structure group via associated shelf

**Theorem (*L.-V. 2015*):** (1) The guitar maps induce a bijective 1-cocycle $J\colon SG_{S,\sigma} \xrightarrow{\sim} SG_{S,\sigma'}$, where $\sigma' = \sigma'_{\triangleleft_\sigma}$.

Reminder: $SG_{S,\sigma} = \langle\, S \mid xy = {}^{x}y x^{y} \,\rangle$.

# Structure group via associated shelf

**Theorem (*L.-V. 2015*):** (1) The guitar maps induce a bijective $\underbrace{\text{1-cocycle}}$ $\boxed{J\colon SG_{S,\sigma} \xrightarrow{\sim} SG_{S,\sigma'}}$, where $\sigma' = \sigma'_{\triangleleft_\sigma}$.

$$J(\overline{x}\overline{y}) = J(\overline{x})^{\overline{y}} J(\overline{y})$$

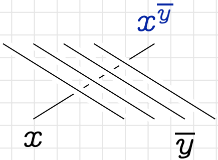
# 8 Structure group via associated shelf

**Theorem (*L.-V. 2015*):** (1) The guitar maps induce a bijective 1-cocycle $J\colon SG_{S,\sigma} \xrightarrow{\sim} SG_{S,\sigma'}$, where $\sigma' = \sigma'_{\lhd_\sigma}$.

$$J(\overline{x}\,\overline{y}) = J(\overline{x})^{\overline{y}} J(\overline{y})$$

$$(x_1, \ldots, x_n)^{\overline{y}} =$$

$$(x_1^{\overline{y}}, \ldots, x_n^{\overline{y}})$$



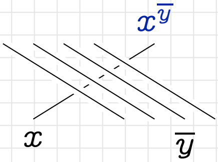
# 8 Structure group via associated shelf

**Theorem (*L.-V. 2015*):** (1) The guitar maps induce a bijective $\underbrace{\text{1-cocycle}}$ $\boxed{J\colon SG_{S,\sigma} \overset{\sim}{\to} SG_{S,\sigma'}}$, where $\sigma' = \sigma'_{\lhd_\sigma}$.

$J(\overline{x}\overline{y}) = J(\overline{x})^{\overline{y}} J(\overline{y})$

$(x_1, \ldots, x_n)^{\overline{y}} =$

$(x_1^{\overline{y}}, \ldots, x_n^{\overline{y}})$




(2) If $(S, \sigma)$ is an RI-compatible birack, then the maps $K^{\times n} J^{(n)}$ induce a bijective 1-cocycle $\boxed{G_{S,\sigma} \overset{\sim}{\to} G_{S,\sigma'}}$,

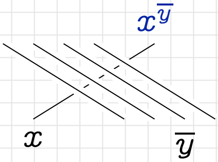
## 8 Structure group via associated shelf

**Theorem (*L.-V. 2015*):** (1) The guitar maps induce a bijective 1-cocycle $J\colon SG_{S,\sigma} \xrightarrow{\sim} SG_{S,\sigma'}$, where $\sigma' = \sigma'_{\lhd_\sigma}$.

$J(\overline{x}\overline{y}) = J(\overline{x})^{\overline{y}} J(\overline{y})$

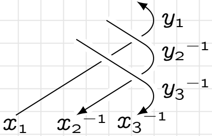
$(x_1, \ldots, x_n)^{\overline{y}} =$

$(x_1^{\overline{y}}, \ldots, x_n^{\overline{y}})$




(2) If $(S, \sigma)$ is an RI-compatible birack, then the maps $K^{\times n} J^{(n)}$ induce a bijective 1-cocycle $G_{S,\sigma} \xrightarrow{\sim} G_{S,\sigma'}$, where

→ $J^{(n)}$ is extended to $(S \sqcup S^{-1})^{\times n}$ by



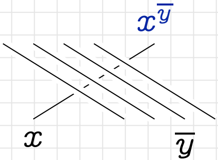
# 8 Structure group via associated shelf

**Theorem (*L.-V. 2015*):** (1) The guitar maps induce a bijective 1-cocycle $J\colon SG_{S,\sigma} \xrightarrow{\sim} SG_{S,\sigma'}$, where $\sigma' = \sigma'_{\triangleleft_\sigma}$.

$J(\overline{x}\overline{y}) = J(\overline{x})^{\overline{y}} J(\overline{y})$

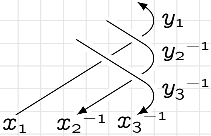
$(x_1, \ldots, x_n)^{\overline{y}} =$

$(x_1^{\overline{y}}, \ldots, x_n^{\overline{y}})$



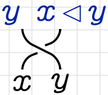

(2) If $(S, \sigma)$ is an RI-compatible birack, then the maps $K^{\times n} J^{(n)}$ induce a bijective 1-cocycle $G_{S,\sigma} \xrightarrow{\sim} G_{S,\sigma'}$, where

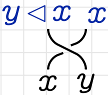
- $J^{(n)}$ is extended to $(S \sqcup S^{-1})^{\times n}$ by
- $K(x) = x$, $K(x^{-1}) = t(x)^{-1}$.



# Associated shelf: examples

✓ For a rack $(S, \triangleleft)$

- $\triangleleft_{\sigma_\triangleleft} = \triangleleft$,
- $J$: 

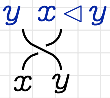
$\leftrightarrow$ 

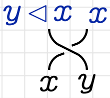
.

# 9 Associated shelf: examples

✓ For a rack $(S, \triangleleft)$

➜ $\triangleleft_{\sigma_\triangleleft} = \triangleleft$,

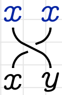
➜ $J$: 

$\leftrightarrow$ 

.

✓ For a group $(S, \star, 1)$

➜ $x \triangleleft_{\sigma_\star} y = y$,

➜ $J$: $\sigma_\star \leftrightarrow$ 

,

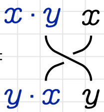
➜ $SG_{S,\sigma'_\star} \xrightarrow{\sim} S$,
$x_1 \cdots x_k \mapsto x_1$.

# 9 Associated shelf: examples

Cycle set: set $S$ & $S \times S \xrightarrow{\cdot} S$ s.t.

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z)$$

& $\forall x,\ y \mapsto x \cdot y$ bijective.

$\Leftrightarrow \sigma_{\cdot} =$ 

is a LND braiding on $S$ with $\sigma_{\cdot}^2 = \mathrm{Id}$

# 9 Associated shelf: examples

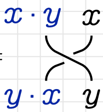
Cycle set: set $S$ & $S \times S \xrightarrow{\cdot} S$ s.t.

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z)$$

& $\forall x,\ y \mapsto x \cdot y$ bijective.

Non-degenerate CS:

& $x \mapsto x \cdot x$ bijective.

$\Leftrightarrow \sigma_{\cdot} =$ 

is a LND braiding on $S$ with $\sigma_{\cdot}^2 = \mathrm{Id}$

$\Leftrightarrow$ birack

# 9 Associated shelf: examples

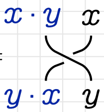
Cycle set: set $S$ & $S \times S \xrightarrow{\cdot} S$ s.t.

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z)$$

& $\forall x,\ y \mapsto x \cdot y$ bijective.

Non-degenerate CS:

& $x \mapsto x \cdot x$ bijective.

$\Leftrightarrow \sigma_{\cdot} =$ 

is a LND braiding on $S$ with $\sigma_{\cdot}^2 = \mathrm{Id}$

$\Leftrightarrow$ birack

(*Etingof–Schedler–Soloviev 1999, Rump 2005*)

# 9 Associated shelf: examples

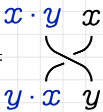
Cycle set: set $S$ & $S \times S \xrightarrow{\cdot} S$ s.t.

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z)$$

& $\forall x,\ y \mapsto x \cdot y$ bijective.

Non-degenerate CS:

& $x \mapsto x \cdot x$ bijective.

$\Leftrightarrow \sigma_{\cdot} =$ 


is a LND braiding on $S$ with $\sigma_{\cdot}^2 = \mathrm{Id}$

$\Leftrightarrow$ birack

(*Etingof–Schedler–Soloviev 1999, Rump 2005*)

Examples: $x \cdot y = f(y)$ for any $f \colon S \xrightarrow{\sim} S$ + glueing.

# 9 Associated shelf: examples

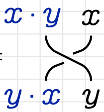
Cycle set: set $S$ & $S \times S \xrightarrow{\cdot} S$ s.t.

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z)$$

& $\forall x$, $y \mapsto x \cdot y$ bijective.

Non-degenerate CS:

& $x \mapsto x \cdot x$ bijective.

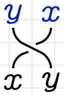
$\Leftrightarrow$ $\sigma_{\cdot} =$ 

is a LND braiding on $S$ with $\sigma_{\cdot}^2 = \mathrm{Id}$

$\Leftrightarrow$ birack

(*Etingof–Schedler–Soloviev 1999, Rump 2005*)

Examples: $x \cdot y = f(y)$ for any $f\colon S \xrightarrow{\sim} S$ + glueing.

- $x \lhd_{\sigma_{\cdot}} y = x$,
- $J\colon \sigma_{\cdot} \leftrightarrow$ flip 

,

# 9 Associated shelf: examples

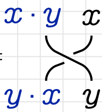
Cycle set: set $S$ & $S \times S \xrightarrow{\cdot} S$ s.t.

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z)$$

& $\forall x$, $y \mapsto x \cdot y$ bijective.

Non-degenerate CS:

& $x \mapsto x \cdot x$ bijective.

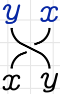
$\Leftrightarrow \sigma_{\cdot} =$ 

is a LND braiding on $S$ with $\sigma_{\cdot}^2 = \mathrm{Id}$

$\Leftrightarrow$ birack

(*Etingof–Schedler–Soloviev 1999, Rump 2005*)

Examples: $x \cdot y = f(y)$ for any $f \colon S \xrightarrow{\sim} S$ + glueing.

- $x \lhd_{\sigma_\cdot} y = x$,
- $J \colon \sigma_{\cdot} \leftrightarrow$ flip 

,
- $(S)G_{S,\sigma_\cdot'}$ is the free abelian (semi)group on $S$.

# 10 Braided (co)homology

(co)homology of $(S, \sigma)$ ⟷ (co)homology of $SG_{S,\sigma}$

small complexes — more tools

(*Carter–Elhamdadi–Saito 2004*)

# 10 Braided (co)homology

| (co)homology of $(S, \sigma)$ | $\longleftrightarrow$ | (co)homology of $SG_{S,\sigma}$ |
|---|---|---|
| small complexes | | more tools |

(*Carter–Elhamdadi–Saito 2004*)

✓ unifies group & rack (co)homologies, and many more;

# 10 Braided (co)homology

(co)homology of $(S, \sigma)$ $\longleftrightarrow$ (co)homology of $SG_{S,\sigma}$

small complexes — more tools

(*Carter–Elhamdadi–Saito 2004*)

✓ unifies group & rack (co)homologies, and many more;

✓ a new theory for cycle sets and braces (*L.-V. 2015, 2016*);

# 10 Braided (co)homology

| (co)homology of $(S,\sigma)$ | $\longleftrightarrow$ | (co)homology of $SG_{S,\sigma}$ |
|---|---|---|
| small complexes | | more tools |

(*Carter–Elhamdadi–Saito 2004*)

✓ unifies group & rack (co)homologies, and many more;

✓ a new theory for cycle sets and braces (*L.-V. 2015, 2016*);

✓ graphical calculus;

✓ a lot of structure (cup product etc.; *Farinati– García-Galofre 2016*);

# 10 Braided (co)homology

| (co)homology of $(S, \sigma)$ | $\longleftrightarrow$ | (co)homology of $SG_{S,\sigma}$ |
| --- | --- | --- |
| small complexes | | more tools |

(*Carter–Elhamdadi–Saito 2004*)

✓ unifies group & rack (co)homologies, and many more;

✓ a new theory for cycle sets and braces (*L.-V. 2015, 2016*);

✓ graphical calculus;

✓ a lot of structure (cup product etc.; *Farinati– García-Galofre 2016*);

✓ for LND braidings, a simpler form via the guitar map (*L.-V. 2015*);

# 10 Braided (co)homology

| (co)homology of $(S,\sigma)$ | $\longleftrightarrow$ | (co)homology of $SG_{S,\sigma}$ |
|---|---|---|
| small complexes | | more tools |

(*Carter–Elhamdadi–Saito 2004*)

✓ unifies group & rack (co)homologies, and many more;

✓ a new theory for cycle sets and braces (*L.-V. 2015, 2016*);

✓ graphical calculus;

✓ a lot of structure (cup product etc.; *Farinati– García-Galofre 2016*);

✓ for LND braidings, a simpler form via the guitar map (*L.-V. 2015*);

✓ applications to the computation of group and Hochschild (co)homology for factorizable groups and for Young tableaux (*L. 2016*).