Applications of self-distributivity to Yang–Baxter operators and their cohomology

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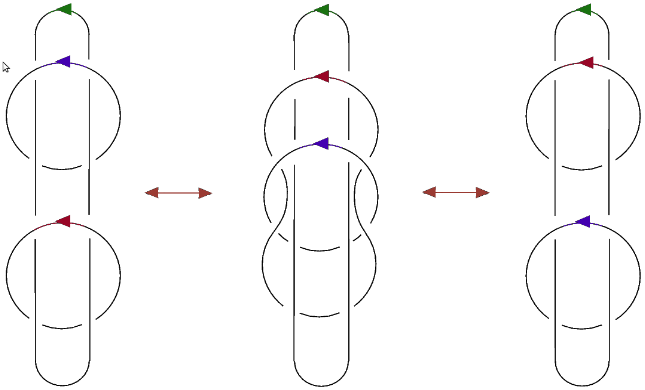
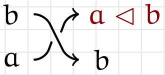
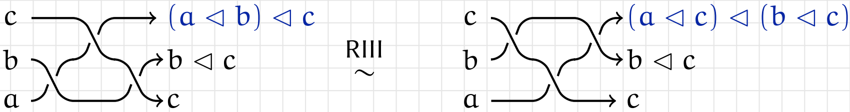


1 Coloring invariants for braids

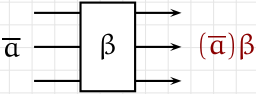
Self-distributivity: $(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$

Diagram colorings by $(S, \lhd)$
for positive braids:





$\mathrm{End}(S^n) \leftarrow B_n^+$	RIII	$(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$



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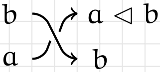

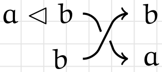

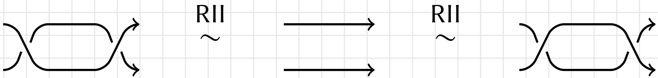

$\mathrm{End}(S^n) \leftarrow B_n^+$	RIII	$(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$	shelf
$\mathrm{Aut}(S^n) \leftarrow B_n$	& RII	$\forall b,\ a \mapsto a \lhd b$ invertible	rack
$S \hookrightarrow (S^n)^{B_n}$		$a \lhd a = a$	quandle

$a \mapsto (a, \ldots, a)$

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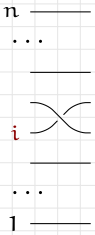
Examples:

S	$a \lhd b$	$(S, \lhd)$ is a	in braid theory
${}_{\mathbb{Z}[t^{\pm 1}]}\mathrm{Mod}$	$ta + (1-t)b$	quandle	(red.) Burau: $B_n \to GL_n(\mathbb{Z}[t^{\pm}])$

$\rho_B($ 

$) = I_{i-1} \oplus \begin{pmatrix} 1-t & 1 \\ t & 0 \end{pmatrix} \oplus I_{n-i-1}$

1 Coloring invariants for braids

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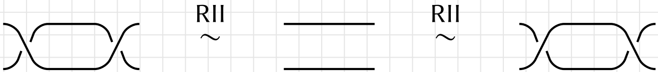
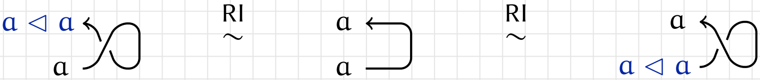
group	$b^{-1}ab$	quandle	Artin: $B_n \hookrightarrow \mathrm{Aut}(F_n)$
twisted linear quandle			Lawrence–Krammer–Bigelow
$\mathbb{Z}$	$a+1$	rack	$\lg(w), \mathrm{lk}_{i,j}$
free shelf			Dehornoy: order on B_n
Laver tables			???

2 Coloring counting invariants for knots

Diagram colorings by $(S, \triangleleft)$ for knots:





pos. braids	RIII	$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$	shelf
braids	& RII	$\forall b,\ a \mapsto a \triangleleft b$ invertible	rack
knots & links	& RI	$a \triangleleft a = a$	quandle

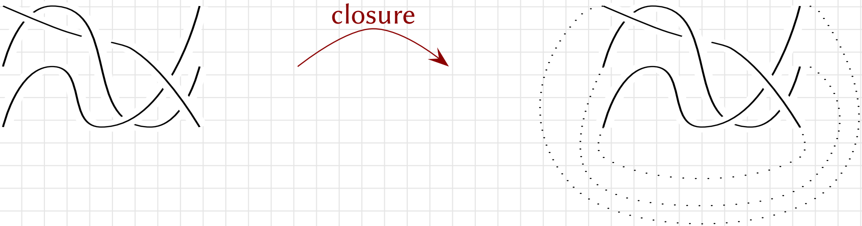
2 Coloring counting invariants for knots

Theorem (*Joyce & Matveev* '82):

✓ The number of colorings of a diagram D of a knot K by a quandle $(S, \triangleleft)$ yields a knot invariant.

✓ $\#\,\mathrm{Col}_{S,\triangleleft}(D) = \#\,\mathrm{Hom}_{\mathsf{Quandle}}(Q(K), S) = \mathrm{Tr}(\rho_S(\beta))$

- $Q(K)$ = fundamental quandle of K
  (a weak universal knot invariant);
- $\mathrm{closure}(\beta) = K$;
- $\rho_S \colon B_n \to \mathrm{Aut}(S^n)$ is the S-coloring invariant for braids.



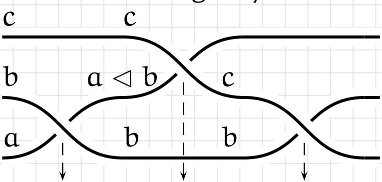
3 Enhancing invariants: weights

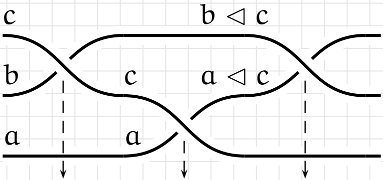
Fenn–Rourke–Sanderson ’95 & *Carter–Jelsovsky–Kamada–Langford–Saito* ’03:

Shelf S, $\phi\colon S \times S \to \mathbb{Z}_n$ $\quad\rightsquigarrow\quad$ ϕ-weights:

$$S\text{-colored diagram } D \quad \longmapsto \quad \sum \pm\phi(a,b)$$

The multi-set of weights yields a braid invariant iff





$$\phi(a,b)+\phi(a\lhd b,c)+\cancel{\phi(b,c)} \;=\; \cancel{\phi(b,c)}+\phi(a,c)+\phi(a\lhd c,b\lhd c)$$

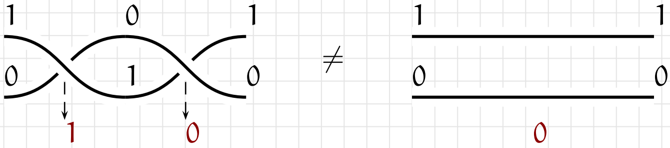
and a knot invariant if moreover $\phi(a,a)=0$.

3 Enhancing invariants: weights

These ϕ-weights strengthen coloring invariants.

Example: $S = \{0, 1\}$, $a \triangleleft b = a$,

$\phi(0, 1) = 1$ and $\phi(a, b) = 0$ elsewhere.



Conjecture (*Clark–Saito–…*):

Finite quandle cocycle invariants distinguish all knots.

More generally, this approach works for knottings $K^{n-1} \hookrightarrow \mathbb{R}^{n+1}$.

4 Self-distributive cohomology

$$C_R^k(S, \mathbb{Z}_n) = \mathrm{Map}(S^{\times k}, \mathbb{Z}_n),$$

$$(d_R^k f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} (f(a_1, \ldots, \widehat{a_i}, \ldots, a_{k+1}) - f(a_1 \lhd a_i, \ldots, a_{i-1} \lhd a_i, a_{i+1}, \ldots, a_{k+1}))$$

$\rightsquigarrow$ Rack cohomology $H_R^k(S, \mathbb{Z}_n)$.

Applications:

① (Higher) braid and knot invariants:

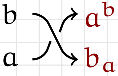
$d_R^2 \phi = 0 \implies \phi$ refines (positive) braid coloring invariants,
$\phi = d_R^1 \psi \implies$ the refinement is trivial.

② Hopf algebra classification (*Andruskiewitsch–Graña* '03).

③ Rack/quandle extensions, deformations etc.

5 Upper strands matter

Diagram colorings by (S, σ):

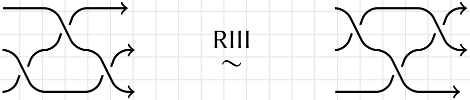


$\sigma(a, b) = (b_a, a^b)$

Ex.: $\sigma_{\triangleleft}(a, b) = (b, a \triangleleft b)$

RIII-compatibility $\iff$ set-theoretic Yang-Baxter equation:

$$\boxed{\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \colon S^{\times 3} \to S^{\times 3}} \qquad \sigma_1 = \sigma \times \mathrm{Id}_S,\ \sigma_2 = \mathrm{Id}_S \times \sigma$$



Set-theoretic solutions $\overset{\text{linearize}}{\rightsquigarrow}$ $\overset{\text{deform}}{\rightsquigarrow}$ linear solutions.

Example: $\sigma(a, b) = (b, a)$ $\rightsquigarrow$ R-matrices.

5 Upper strands matter

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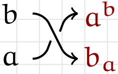

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Exotic example: $\sigma(a, b) = (b, a)$ ⇝

$\sigma_{Lie}(a \otimes b) = b \otimes a + \hbar 1 \otimes [a, b]$, where $[1, a] = [a, 1] = 0$:

YBE for σ_{Lie} $\iff$ Leibniz relation for $[\,]$

Very exotic example: $\sigma_{Ass}(a, b) = (a * b, 1)$, where $1 * a = a$:

YBE for σ_{Ass} $\iff$ associativity for $*$

5 Upper strands matter

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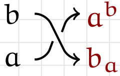


$\sigma(a, b) = (b_a, a^b)$

Ex.: $\sigma_{\triangleleft}(a, b) = (b, a \triangleleft b)$

RIII	$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$	YB operator
& RII	σ invertible & $\forall b,\ a \mapsto a^b$ and $a \mapsto a_b$ invertible	birack
& RI	$\exists$ a bijection t such that $\sigma(t(a), a) = (t(a), a)$	biquandle

Result: Coloring invariants of braids and knots.

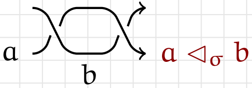
Bad news: These invariants give nothing new!

Unrelated question: Describe free biracks and biquandles.

6 From biracks to racks

Thm (*Soloviev & Lu–Yan–Zhu* '00, *L.–Vendramin* '17):

✓ Birack (S, σ) $\rightsquigarrow$ its structure rack $(S, \lhd_\sigma)$:



✓ This is a projection **Birack** $\twoheadrightarrow$ **Rack** along involutive biracks:

- $\lhd_{\sigma_\lhd} = \lhd$;
- $\lhd_\sigma$ trivial $\iff$ $\sigma^2 = \mathrm{Id}$.

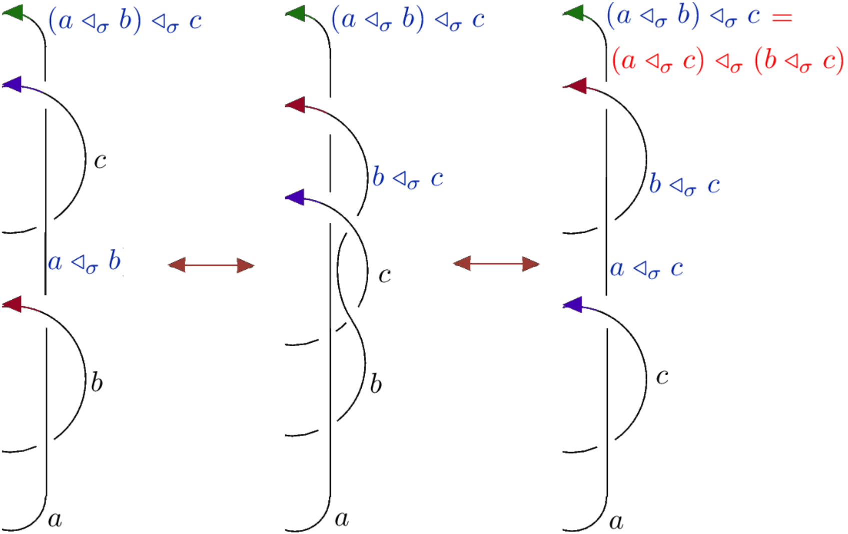
✓ The structure rack remembers a lot about the birack:

- $(S, \lhd_\sigma)$ quandle $\iff$ (S, σ) biquandle;
- σ and $\lhd_\sigma$ induce isomorphic B_n-actions on S^n

$\implies$ same braid and knot invariants.

6 From biracks to racks

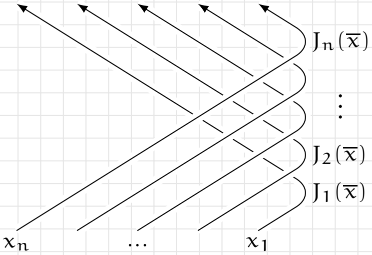
Operation $\lhd_\sigma$ is self-distributive:

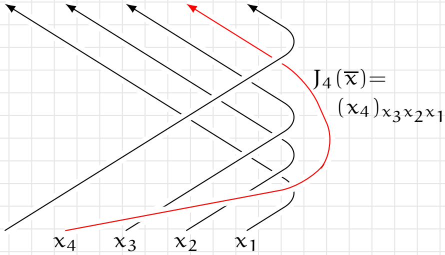


7 Guitar map

$$J\colon S^{\times n} \xrightarrow{1:1} S^{\times n},$$

$$(x_n, \dots, x_1) \longmapsto (\dots, (x_3)_{x_2 x_1}, (x_2)_{x_1}, x_1).$$





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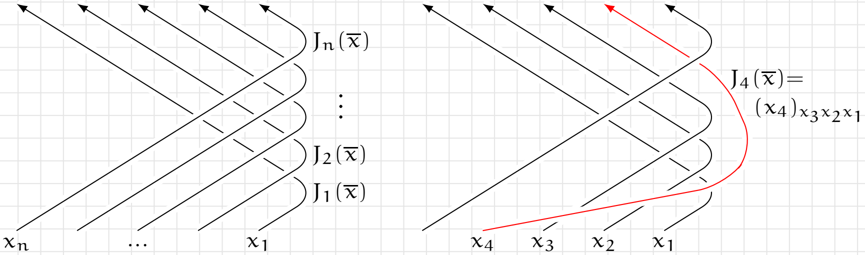

Ex.: $\sigma_{Ass}(a, b) = (ab, 1) \quad \rightsquigarrow \quad J(a, b, c) = (a, ab, abc).$

Ex.: $\sigma_{SD}(a, b) = (b \triangleleft a, a) \quad \rightsquigarrow \quad J(a, b, c) = (a, b \triangleleft a, (c \triangleleft b) \triangleleft a).$

Ex.: $\sigma^2 = \mathrm{Id} \quad \rightsquigarrow \quad \Omega$ from right-cyclic calculus.

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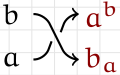
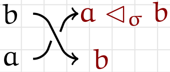
$$J\colon S^{\times n} \xrightarrow{1:1} S^{\times n},$$
$$(x_n, \ldots, x_1) \longmapsto (\ldots, (x_3)_{x_2 x_1}, (x_2)_{x_1}, x_1).$$

Proposition: $\boxed{J\sigma_i = \sigma_i' J}$.

σ: 

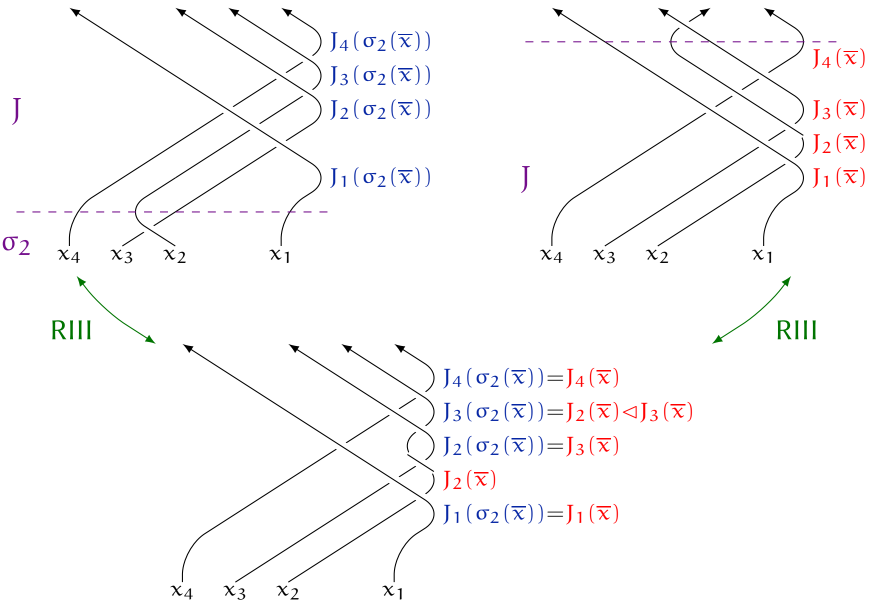
σ': 

Corollary: Same B_n-actions and knot invariants.

⚠ $(S, \sigma) \ncong (S, \sigma')$ as biracks!

Proposition: $\boxed{J\sigma_i = \sigma_i' J}$.

Proof:



8 Braided cohomology

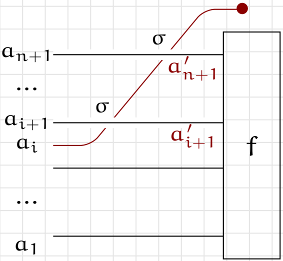
Carter–Elhamdadi–Saito '04 & *L.* '13:

$$C^k_{Br}(S, \mathbb{Z}_n) = \mathrm{Map}(S^{\times k}, \mathbb{Z}_n),$$

$$(d^k_{Br} f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} \big(f(a_1, \ldots, a_{i-1}, (a_{i+1}, \ldots, a_{k+1})_{a_i})$$

$$- f((a_1, \ldots, a_{i-1})^{a_i}, a_{i+1}, \ldots, a_{k+1})\big)$$

$$= \sum (-1)^{i-1} \Big($$



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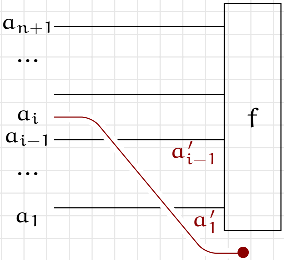


$\Big)$

$\rightsquigarrow$ Braided cohomology $H^k_{Br}(S, \mathbb{Z}_n)$.

9 Why I like braided cohomology

① (Higher) braid and knot invariants:

$$d^2_{\mathrm{Br}}\phi = 0 \implies \phi \text{ refines (positive) braid coloring invariants,}$$
$$\phi = d^1_{\mathrm{Br}}\psi \implies \text{the refinement is trivial.}$$

Question: New invariants?

Answer: I don't know!

② $d^2_{\mathrm{Br}}\phi = 0 \implies$ diagonal deformations of σ:

$$\sigma_q(a, b) = q^{\phi(a,b)}\sigma(a, b).$$

(*Freyd–Yetter* '89, *Eisermann* '05)

9 Why I like braided cohomology

③ Unifies cohomology theories for

✓ self-distributive structures $\sigma_{SD}(a, b) = (b \triangleleft a, a)$

✓ associative structures $\sigma_{Ass}(a, b) = (a * b, 1)$

✓ Lie algebras $\sigma_{Lie}(a \otimes b) = b \otimes a + \hbar 1 \otimes [a, b]$

....

+ explains parallels between them,

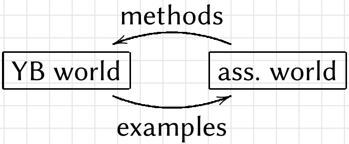
+ suggests theories for new structures.

9 Why I like braided cohomology

④ For certain σ, computes the group cohomology of

$$\boxed{\mathrm{Grp}(S,\sigma) = \langle\, S \mid ab = b_a\, a^b, \text{ where } \sigma(a,b) = (b_a\,, a^b)\,\rangle}$$

Example: $\mathrm{Grp}(S,\sigma_{SD}) = \langle\, S \mid a\,b = b\,(a \lhd b)\,\rangle = \mathrm{As}(S,\lhd)$.

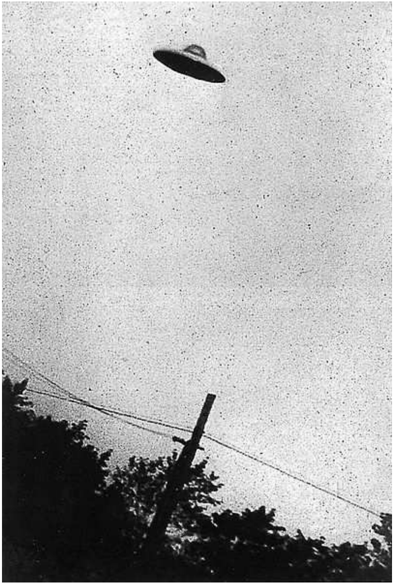


Applications: Cohomology of factorized groups & plactic monoids.

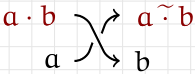
Rmk: $\mathrm{Grp}(S,\sigma)$-modules are coefficients for braided cohomology ("walls").

Rmk: Structure racks know a lot about structure groups.

10 Flying saucer cohomology



10 Flying saucer cohomology

Sideways maps: 

Fenn–Rourke–Sanderson ’93, *Ceniceros–Elhamdadi–Green–Nelson* ’14:

$$C^k_{\text{Bir}}(S, \mathbb{Z}_n) = \text{Map}(S^{\times k}, \mathbb{Z}_n),$$

$$(d^k_{\text{Bir}} f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} \big(f(a_1, \ldots, \widehat{a_i}, \ldots, a_{k+1})$$
$$- f(a_i \tilde{\cdot} a_1, \ldots, a_i \tilde{\cdot} a_{i-1}, a_i \cdot a_{i+1}, \ldots, a_i \cdot a_{k+1})\big)$$

$\rightsquigarrow$ Birack cohomology $H^k_{\text{Bir}}(S, \mathbb{Z}_n)$.

Normalized subcomplex C^k_N for biquandles: $f(\ldots, a_i, a_i, \ldots) = 0$.

Application: Braid and knot invariants.

10 Guitar map counter-attacks

Thm (*L.–Vendramin* '17):

- ✓ Braided and birack cohomologies are the same:

$$J^*\colon (C^\bullet_{Bir}(S,\mathbb{Z}_n), d^\bullet_{Bir}) \cong (C^\bullet_{Br}(S,\mathbb{Z}_n), d^\bullet_{Br}).$$

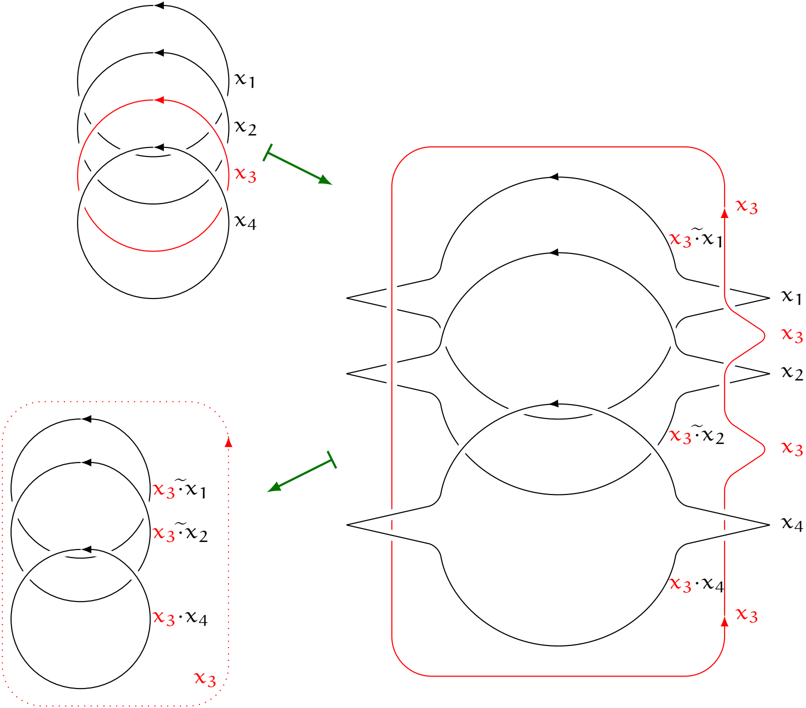
- ✓ For biquandles, cohomology decomposes: $C^\bullet_{Bir} \cong C^\bullet_N \oplus C^\bullet_D$.

Question: Does $C^\bullet_N$ determine $C^\bullet_D$?

Particular cases:

- ✓ a rack $(X,\lhd)$ and its dual $(X,\widetilde{\lhd})$ have the same cohomology (folklore);
- ✓ cohomology decomposition for quandles (*Litherland–Nelson* '03);
- ✓ two forms of group cohomology (folklore);
- ✓ new results for involutive biracks.

Proof: Use a graphical version of d^*_{Bir} & play with diagrams!



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