How forgetting group laws leads to a universal knot invariant

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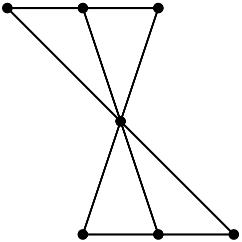
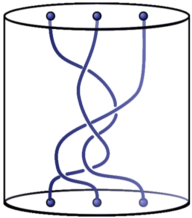
CDMX, May 2017

$g \lhd h = h^{-1}gh$

$0, 1, 2, 3, \dots;$
$\aleph_0, \aleph_1, \aleph_2, \dots;$
$\aleph_\omega, \dots$





1 An exotic axiom for classical structures

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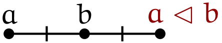
Motivation: geometric symmetries.



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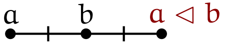
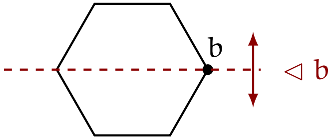


General construction: Abelian group A with $a \lhd b = 2b - a$.

One more geometric example: $\mathbb{Z}_n$.



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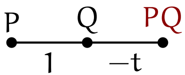

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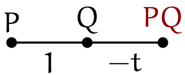

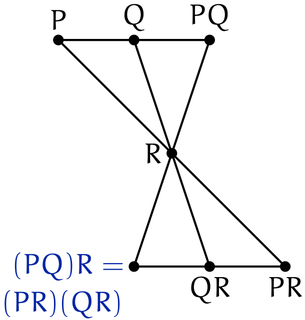

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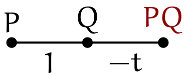

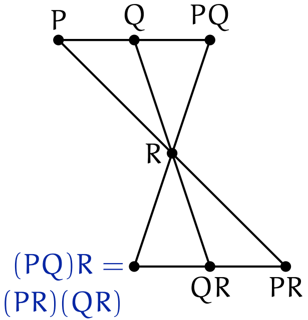

Game 2: Any group (example: S_5) with $g \lhd h = h^{-1}gh$.

2 From curiosity to a theory

Self-distributivity: $(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$

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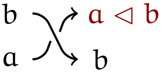
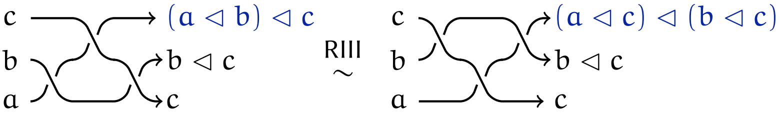
Diagram colourings by $(S, \lhd)$
for positive braids:





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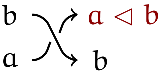
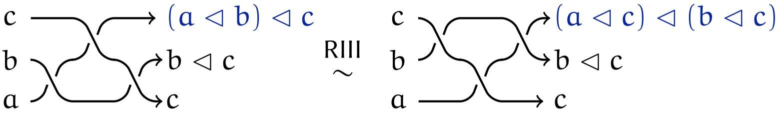
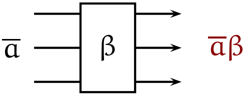




$\mathrm{End}(S^n) \leftarrow B_n^+$	RIII	$(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$



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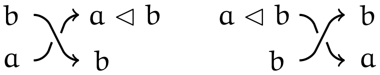

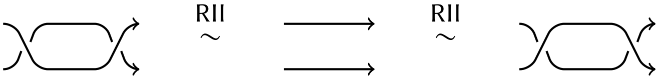

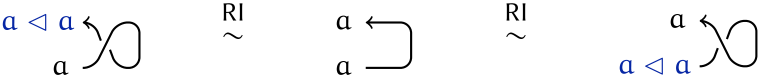

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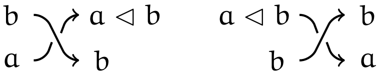
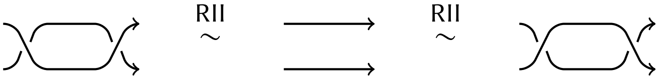
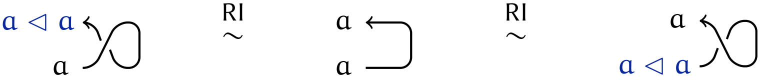


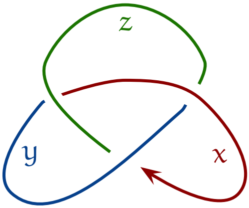


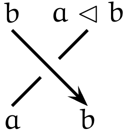


pos. braids	RIII	$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$	shelf
braids	& RII	$\forall b,\ a \mapsto a \triangleleft b$ invertible	rack
knots & links	& RI	$a \triangleleft a = a$	quandle

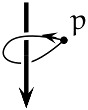
3 Knot invariants: color and conquer

Colourings of knot diagrams by a quandle $(S, \triangleleft)$:





cf. Wirtinger presentation of $\pi_1(\mathbb{R}^3 \setminus K)$:



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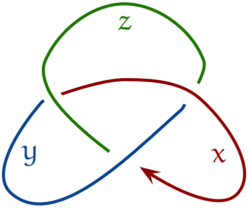

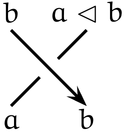

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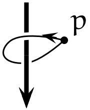

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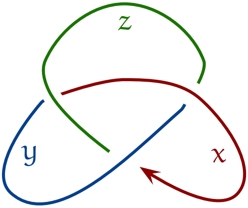

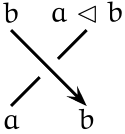

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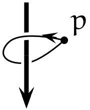

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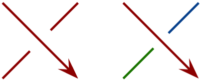

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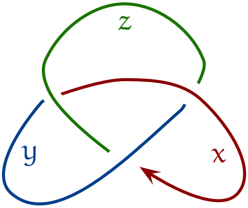

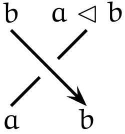

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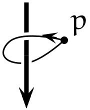

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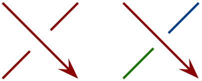

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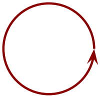

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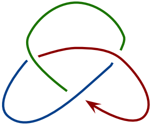

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✓ Easy to program.

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$$K \longmapsto Q(K): \quad \text{generators} \leftrightarrow \text{arcs of } D_K$$

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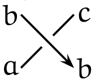

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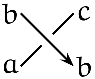

In particular, the fundamental quandle $Q(K)$

✦ does not depend on the choice of a diagram D_K of K;

✦ is a weak universal knot invariant.

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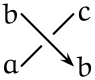

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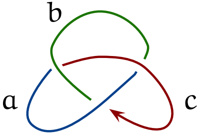

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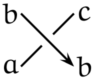

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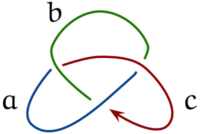

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Remark: Colourings = representations:

$$\mathrm{Col}_S(D_K) \leftrightarrow \mathrm{Hom}_{Qu}(Q(K), S).$$

5 Colouring invariants for braids

$\mathrm{End}(S^n) \leftarrow B_n^+$	$(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$	shelf
$\mathrm{Aut}(S^n) \leftarrow B_n$	$(a \lhd b) \mathbin{\widetilde{\lhd}} b = a = (a \mathbin{\widetilde{\lhd}} b) \lhd b$	rack
$S \hookrightarrow (S^n)^{B_n}$	$a \lhd a = a$	quandle

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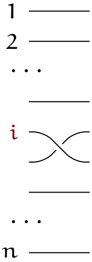
Examples:

S	$a \lhd b$	$(S, \lhd)$ is a	in braid theory
${}_{\mathbb{Z}[t^{\pm 1}]}\mathrm{Mod}$	$ta + (1-t)b$	quandle	(red.) Burau: $B_n \to GL_n(\mathbb{Z}[t^{\pm}])$

$\rho_B($



$) = I_{i-1} \oplus \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}$

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Remark: Quandles are "racks and ruins of groups":

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- ✓ Quandle axioms = conjugation axioms.

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free shelf			Dehornoy: order on B_n

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Remark: Free shelves are extremely rich objects!

6 Building bridges

Self-distributivity: $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$

④ *Richard Laver & Patrick Dehornoy*, set theorists hiding from the I3 axiom.

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$\Longrightarrow$ Discovery of new shelves from

✓ topology: braids;

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$$\gamma = 1 \qquad (\gamma \triangleright \gamma) \triangleright \gamma = 3$$

$$\gamma \triangleright \gamma = 2 \qquad ((\gamma \triangleright \gamma) \triangleright \gamma) \triangleright \gamma = 4 \qquad \ldots$$

7 The I3 axiom counter-attacks

Elementary definition: $A_n = (\{1, 2, 3, \ldots, 2^n\}, \triangleright)$ s.t.

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Some of the **elementary properties:**

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✓ $A_n \rightsquigarrow$ all finite monogenic shelves (*Drápal* '97).

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
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Elementary conjectures:

✓ $\pi_n(1) \underset{n\to\infty}{\to} \infty$.

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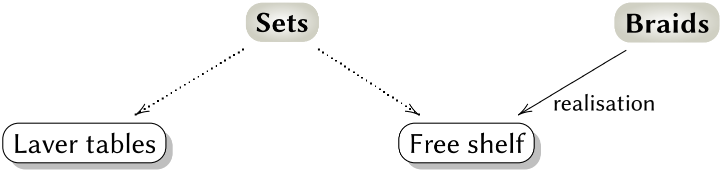

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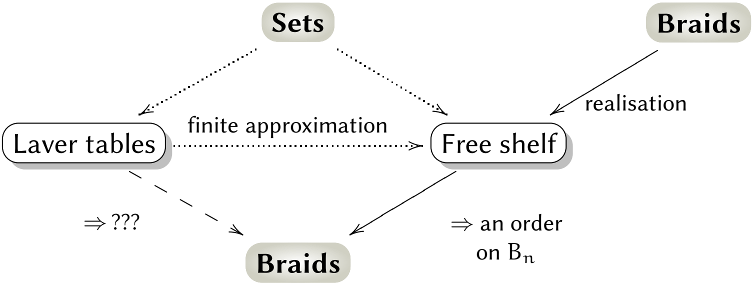
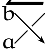


8 Getting more out of colourings

⑤ *Fenn–Rourke–Sanderson & Carter–Jelsovsky–Kamada–Langford–Saito*, refined knot colourists.

Shelf S, $\phi\colon S \times S \to \mathbb{Z}_n$ $\quad\rightsquigarrow\quad$ ϕ-weights:

$$S\text{-coloured diagram } D \quad \longmapsto \quad \sum \pm\phi(a,b)$$



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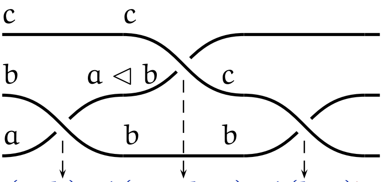
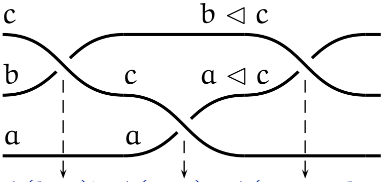
This is an invariant of coloured diagrams iff





$\phi(a, b) + \phi(a \triangleleft b, c) + \cancel{\phi(b, c)} = \cancel{\phi(b, c)} + \phi(a, c) + \phi(a \triangleleft c, b \triangleleft c)$

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Thm: one has a cochain complex $(\mathrm{Map}(S^{\times k}, \mathbb{Z}_n), d_R^k)$,

$$(d_R^k f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} (f(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{k+1}) - f(a_1 \triangleleft a_i, \ldots, a_{i-1} \triangleleft a_i, a_{i+1}, \ldots, a_{k+1})).$$

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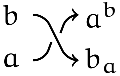
Another **application**: pointed Hopf algebra classification
(*Andruskiewitsch–Graña* '03).

10 Getting symmetric

Diagram colourings by (S, σ):



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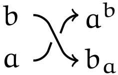

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RIII-compatibility $\iff$ set-theoretic Yang-Baxter equation:

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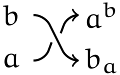

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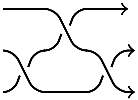
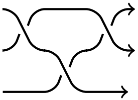


$\overset{\text{RIII}}{\sim}$



⑥ *Vladimir Drinfel′d*, a divider-and-conqueror.

Set-theoretic solutions $\overset{\text{linearise}}{\rightsquigarrow}$ $\overset{\text{deform}}{\rightsquigarrow}$ linear solutions.

Example: $\sigma(x,y) = (y,x)$ $\rightsquigarrow$ R-matrices.

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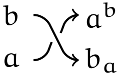

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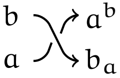

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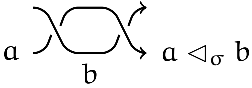

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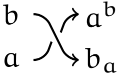

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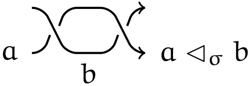

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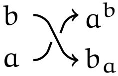

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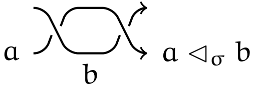

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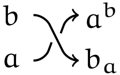

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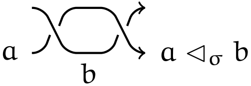

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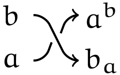

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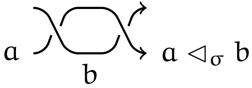

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Here "weight" = a ϕ-weight for a braided 2-cocycle ϕ.

Vote for braided cohomology!

Reasons:

1) $d^2_{\mathrm{Br}}\phi = 0 \implies$ ϕ refines (positive) braid colouring invariants.
$\phi = d^1_{\mathrm{Br}}\psi \implies$ the refinement is trivial.

(*Carter–Elhamdadi–Saito* '04)

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3) Unifies cohomology theories for
   - ✓ self-distributive structures,
   - ✓ associative structures,
   - ✓ Lie algebras etc.

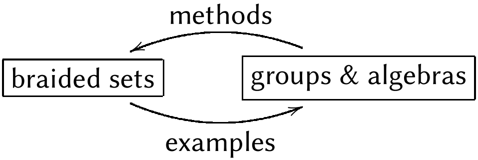
+ explains parallels between them,

+ suggests theories for new structures.

11 Vote for braided cohomology!

4) For certain σ, computes the Hochschild cohomology of

$$\mathrm{Mon}(S,\sigma) = \langle\, S \mid ab = b'a' \text{ whenever } \sigma(a,b) = (b',a') \,\rangle$$



11 Vote for braided cohomology!

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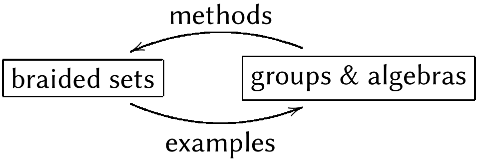

Thm: $\sigma^2 = \mathrm{Id}$ & … $\Longrightarrow$

- ✓ $\mathrm{Mon}(S,\sigma)$ is of I-type, cancellative, Ore;
- ✓ $\mathrm{Grp}(S,\sigma)$ is solvable, Garside;
- ✓ $\Bbbk\,\mathrm{Mon}(S,\sigma)$ is Koszul, noetherian, Cohen–Macaulay, Artin–Schelter regular

(*Manin, Gateva-Ivanova & Van den Bergh, Etingof–Schedler–Soloviev, Jespers–Okniński, Chouraqui* …, 80'-…).

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5) Comes with a graphical calculus, based on branched braids.