How braids can help to compute Hochschild cohomology

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Cohomological Methods in Geometry

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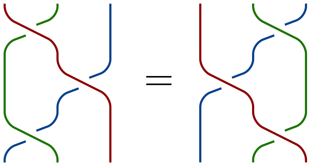


$$(ab)c = a(bc)$$

$$z^{-1}(y^{-1}xy)z = (z^{-1}y^{-1}z)(z^{-1}xz)(z^{-1}yz)$$

1 Yang–Baxter equation: classics

Data: vector space V, $\sigma\colon V^{\otimes 2} \to V^{\otimes 2}$.

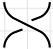
Yang–Baxter equation (YBE)

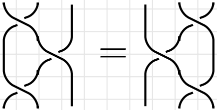
$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\colon V^{\otimes 3} \to V^{\otimes 3}$$

$\sigma_1 = \sigma \otimes \mathrm{Id}_V$, $\sigma_2 = \mathrm{Id}_V \otimes \sigma$

Avatars in:

- statistical mechanics;
- quantum field theory;
- algebra;
- low-dimensional topology:

$\sigma \longleftrightarrow$ 

YBE $\longleftrightarrow$ 

Reidemeister III move

2 First deviation: braided sets

Data: set S, $\sigma\colon S^{\times 2} \to S^{\times 2}$.

Set-theoretic YBE (*Drinfel′d 1990*)

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \colon S^{\times 3} \to S^{\times 3} \qquad \sigma_1 = \sigma \times \mathrm{Id}_S,\ \sigma_2 = \mathrm{Id}_S \times \sigma$$

Solutions are called braided sets.

braided sets $\overset{\text{linearise}}{\rightsquigarrow}$ $\overset{\text{deform}}{\rightsquigarrow}$ general solutions

Examples:

✓ $\sigma(x, y) = (x, y)$;

✓ $\sigma(x, y) = (y, x) \quad \rightsquigarrow \quad$ R-matrices;

✓ Lie algebra $(V, [\,])$, central element $1 \in V$,
$\sigma(x \otimes y) = y \otimes x + \hbar 1 \otimes [x, y]$.

YBE for $\sigma \iff$ Jacobi identity for $[\,]$

3 Self-distributivity

✓ set S, binary operation $\lhd$, $\sigma(x,y) = (y, x \lhd y)$

YBE for σ $\Longleftrightarrow$ self-distributivity for $\lhd$

Self-distributivity: $(x \lhd y) \lhd z = (x \lhd z) \lhd (y \lhd z)$

Examples:

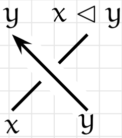
- group S with $x \lhd y = y^{-1}xy$:

$$z^{-1}(y^{-1}xy)z = (z^{-1}y^{-1}z)(z^{-1}xz)(z^{-1}yz)$$

- abelian group S, $t\colon S \to S$, $a \lhd b = ta + (1-t)b$.

Applications:

- invariants of knots and knotted surfaces;
- a total order on braid groups;
- Hopf algebra classification.



4 Even more exotic examples

✓ monoid $(S, *, 1)$, $\sigma(x,y) = (1, x * y)$;

YBE for $\sigma \iff$ associativity for $*$

✓ lattice $(S, \bigwedge, \bigvee)$, $\sigma(x,y) = (x \bigwedge y, x \bigvee y)$.

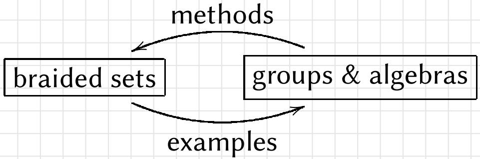
All these braidings are idempotent: $\sigma\sigma = \sigma$.

5 Universal enveloping constructions

Universal enveloping monoids:

$$\mathrm{Mon}(S,\sigma) = \langle\, S \mid xy = y'x' \text{ whenever } \sigma(x,y) = (y',x') \,\rangle$$

U. e. groups and algebras are defined similarly.



Theorem: (S,σ) a “nice” finite braided set, $\sigma^2 = \mathrm{Id} \implies$

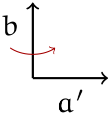
- ✓ $\mathrm{Mon}(S,\sigma)$ is of I-type, cancellative, Ore;
- ✓ $\mathrm{Grp}(S,\sigma)$ is solvable, Garside;
- ✓ $\Bbbk\,\mathrm{Mon}(S,\sigma)$ is Koszul, noetherian, Cohen–Macaulay, Artin–Schelter regular

(*Manin, Gateva-Ivanova & Van den Bergh, Etingof– Schedler–Soloviev, Jespers–Okniński, Chouraqui 80’-…*).

Example: $S = \{a, b\}$, $aa \overset{\sigma}{\longleftrightarrow} bb$

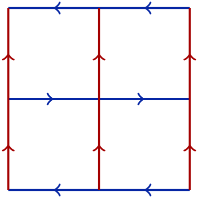
$$\mathrm{Grp}(S, \sigma) = \langle\ a, b \mid a^2 = b^2\ \rangle =: G.$$

Realisation by Euclidean transformations of $\mathbb{R}^2$:



$$b = a'ba'$$
$$\big\downarrow\ a = a'b$$
$$a^2 = b^2$$

$\mathbb{R}^2/G \cong$ Klein bottle:



5 Universal enveloping constructions

Universal enveloping monoids:

$$\mathrm{Mon}(S,\sigma) = \langle\, S \mid xy = y'x' \text{ whenever } \sigma(x,y) = (y',x') \,\rangle$$

Examples:

✓ monoid $(S, *, 1)$, $\sigma(x,y) = (1, x*y)$,

$$S \simeq {}^{\mathrm{Mon}(S,\sigma)}/_{1 = 1_{\mathrm{Mon}}};$$

✓ Lie algebra $(V, [\,], 1)$, $\sigma(x \otimes y) = y \otimes x + \hbar 1 \otimes [x,y]$,

$$\mathrm{UEA}(V, [\,]) \simeq {}^{\Bbbk\,\mathrm{Mon}(V,\sigma)}/_{1 = 1_{\mathrm{Mon}}}.$$

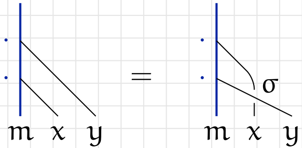
6 Representations

$$\mathrm{Mon}(S,\sigma) = \langle\, S \mid xy = y'x' \text{ whenever } \sigma(x,y) = (y',x') \,\rangle$$

Representations of (S,σ) := representations of $\Bbbk\,\mathrm{Mon}(S,\sigma)$,

i.e. vector spaces M with $M \times S \xrightarrow{\cdot} M$ s.t.

$$(m \cdot x)\cdot y \quad = \quad (m \cdot y')\cdot x'$$



Examples:

- trivial rep.: $M = \Bbbk$, $m \cdot x = m$;
- $M = \Bbbk\,\mathrm{Mon}(S,\sigma)$, $m \cdot x = mx$;
- usual reps for monoids, Lie algebras, self-distributive structures.

7 A cohomology theory?

A cohomology theory for braided sets should:

1) Describe diagonal deformations

$$\sigma_q(x,y) = q^{\omega(x,y)}\sigma(x,y), \ \omega \colon S \times S \to \mathbb{Z} :$$

ω a 2-cocycle $\quad\Longrightarrow\quad$ σ_q a YBE solution.

2) Yield knot and knotted surface invariants:

(S,σ)-coloured diagram $(D,\mathcal{C})$ $\quad\&\quad$ $\omega \colon S \times S \to \mathbb{Z}$

$\rightsquigarrow$ Boltzmann weight $\mathscr{B}_\omega(\mathcal{C}) = \sum\limits_{\substack{y' \nwarrow\ \nearrow x' \\ \times \\ x\ \ \ \ y}} \omega(x,y) - \sum\limits_{\substack{x \nwarrow\ \nearrow y \\ \times \\ y'\ \ \ \ x'}} \omega(x,y).$

ω a 2-cocycle $\quad\Longrightarrow\quad$ a knot invariant given by

$$\{\, \mathscr{B}_\omega(\mathcal{C}) \mid \mathcal{C} \text{ is a } (S,\sigma)\text{-colouring of } D \,\}.$$

7 A cohomology theory?

A cohomology theory for braided sets should:

3) Unify cohomology theories for
  - associative structures,
  - Lie algebras,
  - self-distributive structures etc.

+ explain parallels between them,

+ suggest theories for new structures.

4) Compute the cohomology of $\Bbbk\,\mathrm{Mon}(S, \sigma)$.

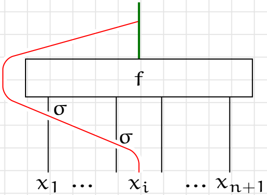
8 Braided cohomology

<u>Data</u>: braided set (S,σ) & bimodule M over it.

<u>Construction</u>:

$C^n(S,\sigma;M) = \mathrm{Maps}(S^{\times n}, M)$,
$d^n = \sum_{i=1}^{n+1}(-1)^{i-1}(d_l^{n;i} - d_r^{n;i})\colon C^n \to C^{n+1}$,

$$d_l^{n;i}f\colon \quad x_1 \dots x_{n+1} \xrightarrow{\sigma_1\cdots\sigma_{i-1}} x'_i x'_1 \dots x'_{i-1} x_{i+1} \dots x_{n+1} \longrightarrow x'_i \cdot f(x'_1 \dots x'_{i-1} x_{i+1} \dots x_{n+1})$$



<u>Theorem</u>:

- $d^{n+1}d^n = 0$;

  $H^*(S,\sigma;M)$ is the <u>braided cohomology</u> of (S,σ) with coefs in M;
- for "nice" M, there is a <u>cup product</u> $\smile\colon H^n \otimes H^m \to H^{n+m}$;
- other good properties.

9 A good theory?

1) & 2) For $\omega \in C^2(S, \sigma; \mathbb{Z})$,

$$d^2\omega = 0 \implies \omega \text{ yields Boltzmann weights}$$
$$\& \text{ diagonal deformations,}$$
$$\omega = d^1\theta \implies \omega \text{ yields trivial...}$$

3) Unifies classical cohomology theories.

Example: monoid $(S, *, 1)$, $\sigma(x, y) = (1, x * y)$,

$$d_l^{n;i}f\colon \quad \ldots x_{i-2}\, \underline{x_{i-1}\, x_i}\, x_{i+1} \ldots \xrightarrow{\sigma_{i-1}}$$
$$\ldots \underline{x_{i-2}\, 1}\, (x_{i-1} * x_i)\, x_{i+1} \ldots \xrightarrow{\sigma_{i-2}}$$
$$\ldots \underline{1} x_{i-2}\, (x_{i-1} * x_i)\, x_{i+1} \ldots \longrightarrow \ldots$$
$$1\, x_1 \ldots x_{i-2}\, (x_{i-1} * x_i)\, x_{i+1} \ldots \longrightarrow$$
$$f(\ldots x_{i-2}\, (x_{i-1} * x_i)\, x_{i+1} \ldots).$$

9 A good theory?

4) Quantum symmetriser $\mathcal{QS}$:

braided cohomology of (S, σ) with coefs in M	$\xleftarrow{\mathcal{QS}}$	Hochschild cohomology of $\Bbbk\,\mathrm{Mon}(S, \sigma)$ with coefs in M
cup product		cup product
smaller complexes		tools

$\mathcal{QS}$ is an isomorphism when

- $\sigma\sigma = \mathrm{Id}$ and $\mathrm{Char}\,\Bbbk = 0$ (*Farinati & García-Galofre 2016*);
- $\sigma\sigma = \sigma$ (*L. 2016*).

Applications:

- factorisable groups,
- Young tableaux.

Open problem: How far is $\mathcal{QS}$ from being an iso in general?