Twisted multi-distributivity and Lawrence representations of braid groups

Victoria LEBED, University of Nantes

Knots in Dallas, January 2015

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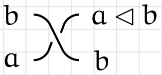
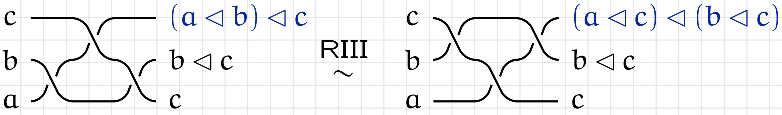
✓ here: a combinatorial version.

2 Braids and self-distributivity

colorings by $(S, \triangleleft)$





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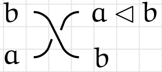

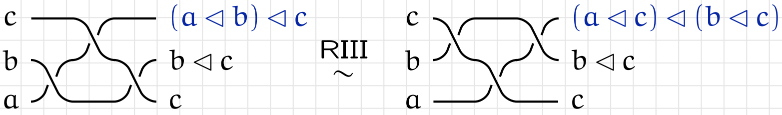

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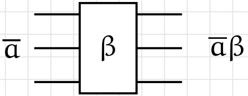

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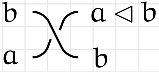

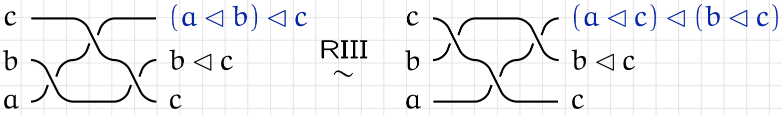

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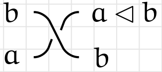

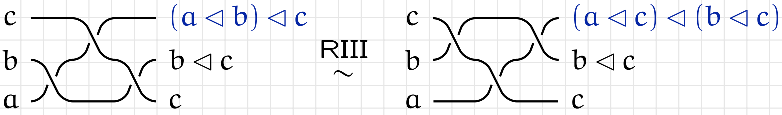

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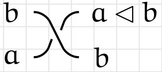

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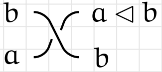

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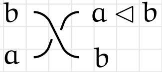

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$\mathbb{Z}$	$a + 1$	rack	$\mathrm{lg}(w), \mathrm{lk}_{i,j}$
free shelf			Dehornoy: order on B_n

3 Twisted multi-distributivity

Fix a group G.

G-quandle: a set S and a family of binary operations $\lhd_g$ on S, $g \in G$, s.t.

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Example: a $\mathbb{Z}G$-module $S \mapsto$ Alexander G-quandle

$$a \lhd_g b = ag + b(1-g)$$

$\rightsquigarrow$ a nice functor $\mathrm{Mod}_{\mathbb{Z}G} \hookrightarrow$ G-Quandles.

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✓ $\Longrightarrow$ all $(S, \lhd_g)$ are quandles;

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✓ G can be replaced with any quandle Q.

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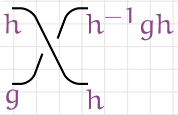

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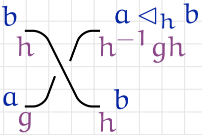

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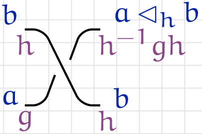

Lemma: compatible with Reidemeister moves.
Remark: works well in the welded (= loop braid) settings.

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G-family of quandles = G-quandle $(S, \lhd_g)$ s.t.

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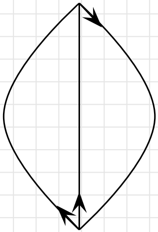
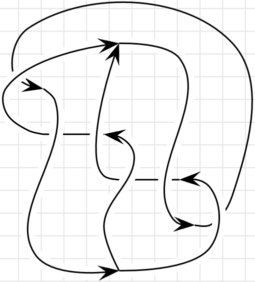


Θ_{st}



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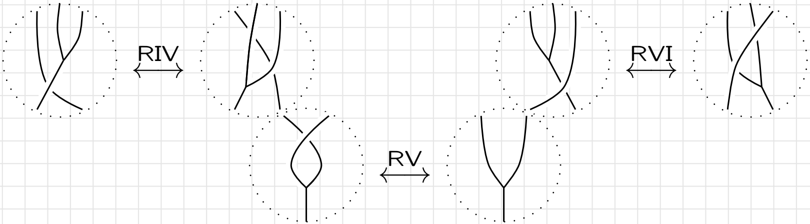

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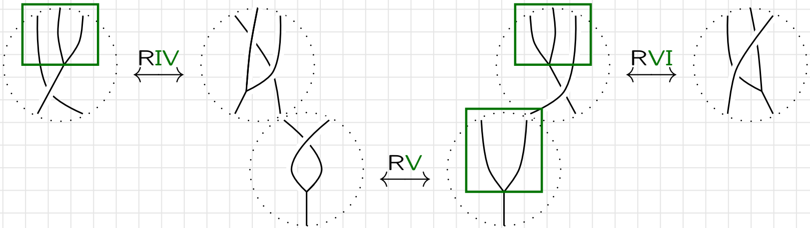

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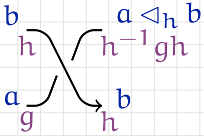

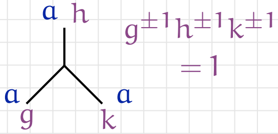

Lemma: Double-layer colorings are compatible with Reidemeister moves.

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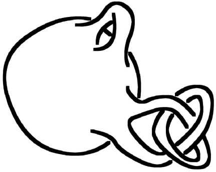
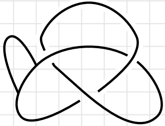
Motivation:

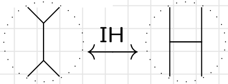
knotted handle-bodies



$\longleftrightarrow$



A. Ishii, 2008: $\cong$ 3-Graphs /



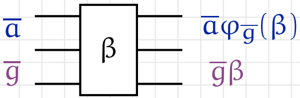
Lemma: Double-layer colorings are compatible with Reidemeister & IH moves

$\rightsquigarrow$ invariants of knotted h-bodies.

5 Quasi-representations of B_n

Take a G-quandle $(S, \triangleleft_g)$.

For any $\overline{g} \in G^n$, one has a map $\varphi_{\overline{g}} \colon B_n \to \mathrm{Aut}(S^n)$ defined by



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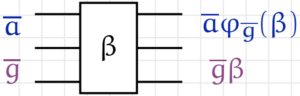

Lemma: $\varphi_{\overline{g}}(\beta\beta') = \varphi_{\overline{g}}(\beta)\varphi_{\overline{g}\beta}(\beta')$.

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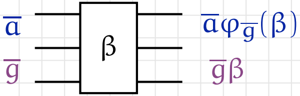

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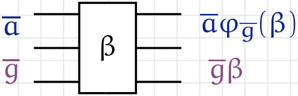

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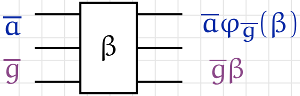
Answer for the particular case

✓ $G = F_n$, $\overline{g}^* = \underbrace{(x_1, \ldots, x_n)}_{\text{generators}} \implies \overline{g}^*\beta = (\beta(x_1), \ldots \beta(x_n))$

✓ S is an Alexander G-quandle $\implies \varphi_{\overline{g}} \colon B_n \to GL_n(\mathbb{Z}G)$

5 Quasi-representations of B_n

Take a G-quandle $(S, \lhd_g)$.
For any $\overline{g} \in G^n$, one has a map $\varphi_{\overline{g}}\colon B_n \to \mathrm{Aut}(S^n)$ defined by



Lemma: $\varphi_{\overline{g}}(\beta\beta') = \varphi_{\overline{g}}(\beta)\varphi_{\overline{g}\beta}(\beta')$.

Question: Get an honest rep. of B_n?

Answer for the particular case

✓ $G = F_n$, $\overline{g}^* = (\underbrace{x_1, \ldots, x_n}_{\text{generators}})$ $\quad\Longrightarrow\quad$ $\overline{g}^*\beta = (\beta(x_1), \ldots \beta(x_n))$

✓ S is an Alexander G-quandle $\quad\Longrightarrow\quad$ $\varphi_{\overline{g}}\colon B_n \to GL_n(\mathbb{Z}G)$

Theorem: One has a group morphism

$$\varphi\colon B_n \to GL_n(\mathbb{Z}F_n) \rtimes B_n,$$
$$\beta \mapsto (\varphi_{\overline{g}^*}(\beta), \beta).$$

6 Long-Moody construction

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Proof: $(\varphi_{\overline{g}^*}(\beta), \beta)(\varphi_{\overline{g}^*}(\beta'), \beta') = (\varphi_{\overline{g}^*}(\beta) \cdot \beta\varphi_{\overline{g}^*}(\beta'), \beta\beta')$
$= (\varphi_{\overline{g}^*}(\beta)\varphi_{\overline{g}^*\beta}(\beta'), \beta\beta') = (\varphi_{\overline{g}^*}(\beta\beta'), \beta\beta')$.

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Corollary: $\rho\colon F_n \rtimes B_n \to \mathrm{Aut}(V) \quad \overset{\varphi_*}{\rightsquigarrow} \quad \rho^+\colon B_n \to \mathrm{Aut}(V^{\oplus n})$.

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Isn't this construction weird?

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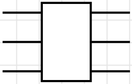
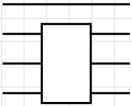
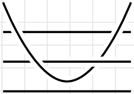
Isn't this construction weird? No!!!

✓ $F_n \rtimes B_n \hookrightarrow B_{n+1}$ in several ways

E.g.  $\mapsto$ 

$x_i \mapsto$  i

$\Longrightarrow$ one can start with a B_{n+1}-rep.

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Examples:

- trivial rep. of B_{n+1} $\rightsquigarrow$ Burau rep. of B_n;

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- trivial rep. of B_2 & scaling & shifting & 2 iterations $\rightsquigarrow$ Lawrence-Krammer rep. of B_n.

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✓ Convenient for explicit calculations:

$$\varphi(\sigma_i) = (\begin{pmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 0 & x_{i+1} & 0 \\ 0 & 1 & 1 - x_{i+1} & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{pmatrix}, \sigma_i),$$

$$\sigma_i x_i = x_{i+1} \sigma_i,\ \sigma_i x_{i+1} = x_{i+1}^{-1} x_i x_{i+1} \sigma_i.$$

7 Reduced version

Corollary: $\rho\colon F_n \rtimes B_n \to \mathrm{Aut}(V) \quad \overset{\varphi_*}{\rightsquigarrow} \quad \rho^+\colon B_n \to \mathrm{Aut}(V^{\oplus n})$.

$$(V, \rho_{B_n}) \hookrightarrow (V^{\oplus n}, \rho^+),$$
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Better: $\rho^+ \simeq \rho_{B_n} \oplus \rho^+_{red}$.

C.f. reduced Burau rep.!

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Question: A self-distributive version of ρ^+_{red}?

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$$(\varphi(\sigma_i) + (g_{i+1}, \sigma_i))(\varphi(\sigma_i) - (1, \sigma_i)) = 0.$$
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  Related constructions of reps of B_n?

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- Other examples of G-quandles ?
  Related constructions of reps of B_n?

- Study emerging "holonomy" Yang-Baxter operators?
  (Cf. Kashaev-Reshetikhin.)