Une promenade dans le livre vert

Victoria LEBED, Université Caen Normandie

Le Havre 2019

1 An exotic axiom for classical structures

Self-distributivity: $\boxed{(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)}$

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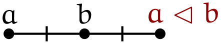
Motivation: geometric symmetries.



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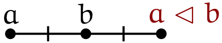

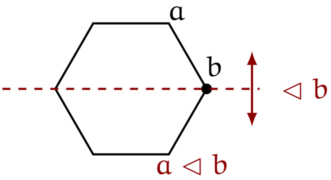

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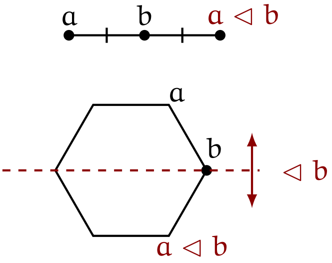

✓ More generally :
abelian group A with $a \lhd b = 2b - a$.

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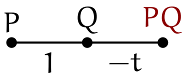

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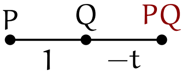

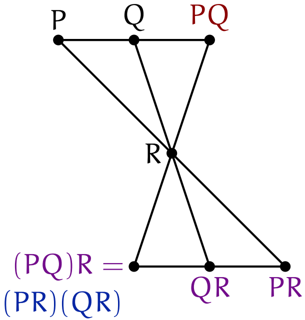

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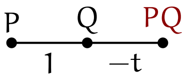

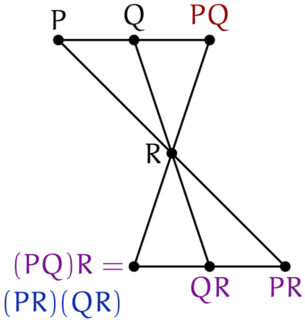

✓ Any group G with $g \lhd h = h^{-1}gh$.

2 From curiosity to a theory

Self-distributivity: $\boxed{(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)}$

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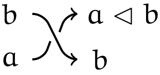
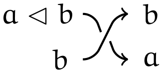
Diagram colorings by $(S, \lhd)$
for braids:





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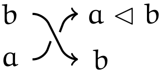
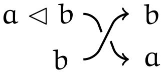
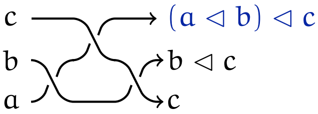
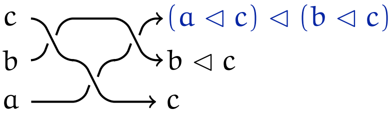






$\overset{\mathrm{RIII}}{\sim}$



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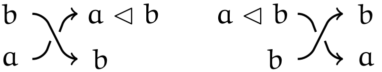
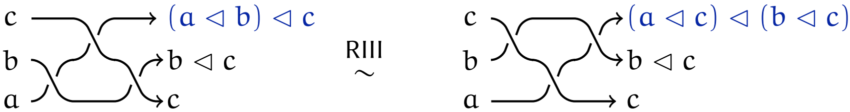
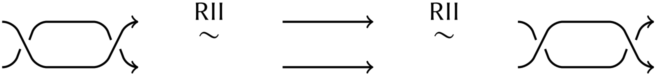
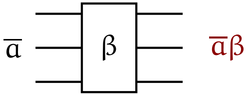






2 From curiosity to a theory

B_n (braid group) acts on S^n	RIII	$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$
	RII	$\forall b, \bullet \triangleleft b$ is invertible



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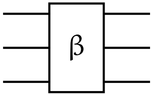
$S \hookrightarrow (S^n)^{B_n}$	(RI)	$a \triangleleft a = a$

$a \mapsto (a, \ldots, a)$

$\overline{a}$



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B_n (braid group) acts on S^n	RIII	$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$	shelf
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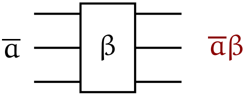

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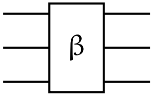
$\overline{a}$  $\overline{a}\beta$

Examples:

S	$a \lhd b$	$(S, \lhd)$ is a	in braid theory
${}_{\mathbb{Z}[t^{\pm 1}]}\mathrm{Mod}$	$ta + (1-t)b$	quandle	Burau: $B_n \to GL_n(\mathbb{Z}[t^{\pm}])$
group	$b^{-1}ab$	quandle	Artin: $B_n \hookrightarrow \mathrm{Aut}(F_n)$
$\mathbb{Z}$	$a+1$	rack	$lg(w)$, $lk_{i,j}$
free shelf $\mathcal{F}_1$			Dehornoy: order on B_n

Progress in Mathematics

Patrick Dehornoy

Braids and Self-Distributivity

INSTITVT D'ESTVDIS CATALANS BARCELONA

Ferran Sunyer i Balaguer
Award winning monograph

Birkhäuser

3 Building bridges

Self-distributivity: $\boxed{(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)}$

④ *Richard Laver & Patrick Dehornoy*, set theorists hiding from the I3 axiom

✓ Elementary embeddings with the application operation.

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Two realisations of the free shelf $\mathcal{F}_1$

1. Elementary embeddings of certain ranks.

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$$\begin{tikzcd} \boxed{\text{free shelf}} \arrow[rr, bend right, "\text{colorings}"'] & & \boxed{\text{braids}} \arrow[ll, bend right, "\text{realisation}"'] \end{tikzcd}$$

4 One more chapter

Laver table $A_n = \{1, 2, 3, \ldots, 2^n\}$ with the unique operation $\triangleright$ satisfying

$$a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright (a \triangleright c) \qquad \& \qquad a \triangleright 1 \equiv a + 1 \mod 2^n.$$

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2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
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Elementary definition and some elementary properties:

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✓ Periodic rows.

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3	4	8	4	8	4	8	4	8	$\pi_3(3) = 2$
4	5	6	7	8	5	6	7	8	$\pi_3(4) = 4$
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5 The I3 axiom counter-attacks

Elementary conjectures:

✓ $\pi_n(1) \underset{n\to\infty}{\to} \infty$.

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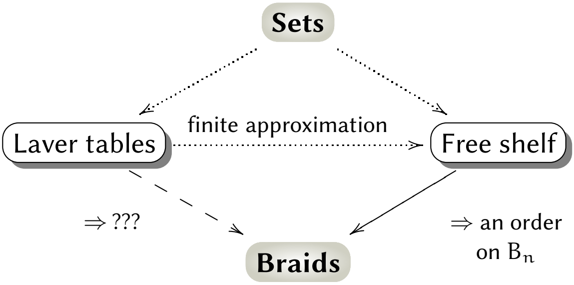
Theorems under the axiom I3!

5 The I3 axiom counter-attacks

Elementary conjectures:

✓ $\pi_n(1) \underset{n\to\infty}{\to} \infty$. ✓ $\pi_n(1) \leqslant \pi_n(2)$. ✓ $\varprojlim_{n\in\mathbb{N}} A_n \supset \mathcal{F}_1$.

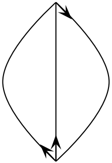
Theorems under the axiom I3!



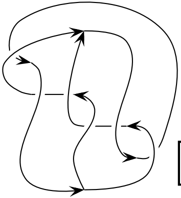
Osaka, Japan

6 Some more topology

Knotted trivalent graphs:



$\boxed{\Theta_{st}}$



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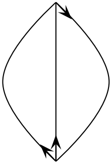

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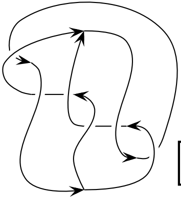
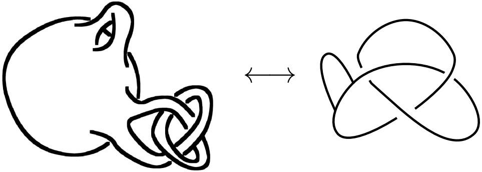


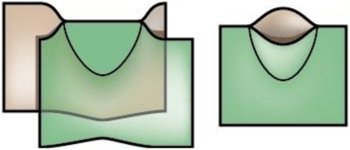
$\boxed{\Theta_{KT}}$

Motivation:

✓ Knotted handle-bodies.



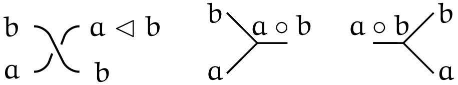
✓ Boundaries of foams.



✓ Form a finitely presented algebraic system (knots do not).

7 A challenge for self-distributivity?

Diagram colorings by $(S, \lhd, \circ)$:



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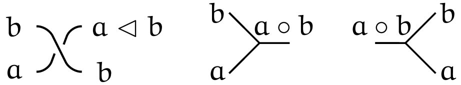
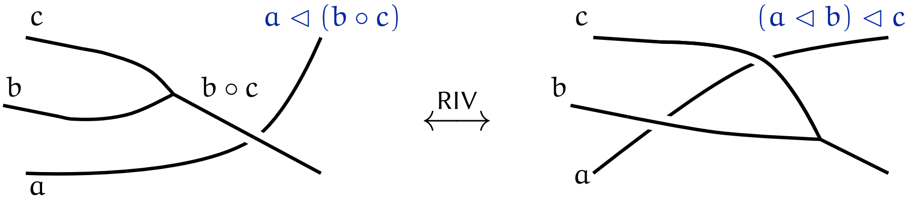


Compatible with topology iff

$$(a \circ b) \triangleleft c = (a \triangleleft c) \circ (b \triangleleft c),$$
$$a \triangleleft (b \circ c) = (a \triangleleft b) \triangleleft c,$$
$$a \circ b = b \circ (a \triangleleft b).$$



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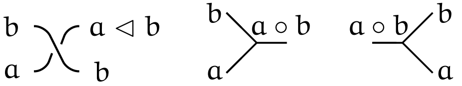
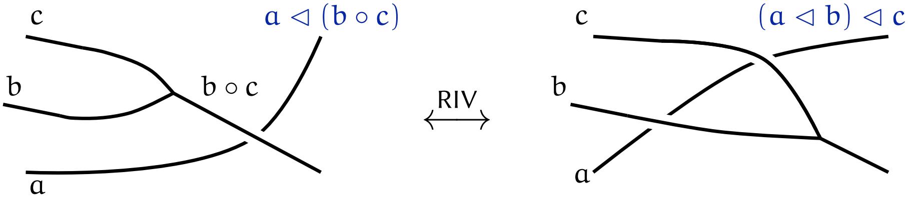


Compatible with topology iff

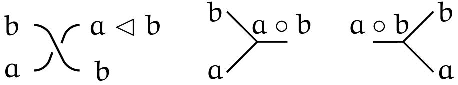
$$(a \circ b) \triangleleft c = (a \triangleleft c) \circ (b \triangleleft c),$$
$$a \triangleleft (b \circ c) = (a \triangleleft b) \triangleleft c,$$
$$a \circ b = b \circ (a \triangleleft b).$$



$\rightsquigarrow$ Powerful invariants of branched braids.

8 Laver tables again!

Diagram colorings by $(S, \triangleleft, \circ)$:



Compatible with topology iff

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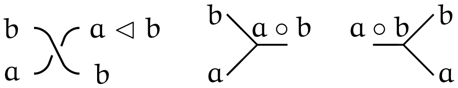
$$a \triangleleft (b \circ c) = (a \triangleleft b) \triangleleft c,$$
$$a \circ b = b \circ (a \triangleleft b).$$

Examples:

✓ group G with $g \triangleleft h = h^{-1}gh,\ g \circ h = gh$;

8 Laver tables again!

Diagram colorings by $(S, \lhd, \circ)$:



Compatible with topology iff

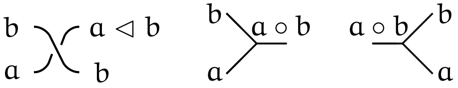
$$
\begin{aligned}
&(a \circ b) \lhd c = (a \lhd c) \circ (b \lhd c), \\
&a \lhd (b \circ c) = (a \lhd b) \lhd c, \\
&a \circ b = b \circ (a \lhd b).
\end{aligned}
$$

Examples:

✓ group G with $g \lhd h = h^{-1}gh,\ g \circ h = gh$;

✓ Laver tables with $p \circ q = p \rhd (q+1) - 1$.

8 Laver tables again!

Diagram colorings by $(S, \lhd, \circ)$: 

Compatible with topology iff

$$
\begin{aligned}
(a \circ b) \lhd c &= (a \lhd c) \circ (b \lhd c),\\
a \lhd (b \circ c) &= (a \lhd b) \lhd c,\\
a \circ b &= b \circ (a \lhd b).
\end{aligned}
$$

Examples:

✓ group G with $g \lhd h = h^{-1}gh,\ g \circ h = gh$;

✓ Laver tables with $p \circ q = p \rhd (q+1) - 1$.

⚠ False for the free shelf $\mathcal{F}_1$!