Journées Normandes en Topologie

21 au 23 octobre 2019 -- Caen

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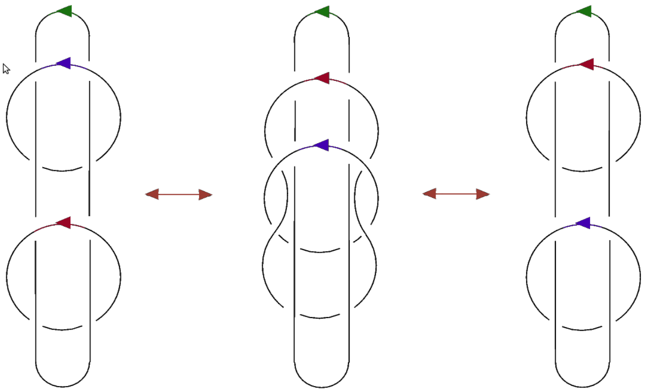
La conférence est organisée par le Laboratoire de mathématiques Nicolas Oresme avec le soutien de GDR Tresses, Université de Caen Normandie, Caen La Mer, Fédération Normandie Mathématiques et en relation avec le projet RIN ARTIQ.

(En cours) Programme et résumés au format pdf.

Braids, biracks, and categorical braidings

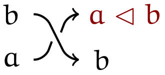
Victoria LEBED, Caen (France)

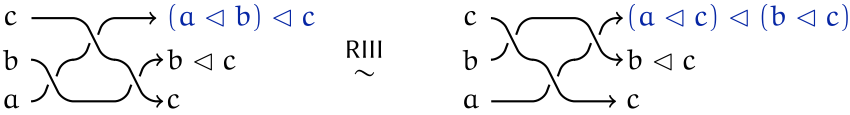
Leeds, July 2019



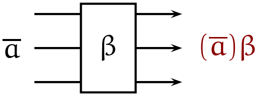
1 Coloring invariants for braids

Diagram colorings by $(S, \triangleleft)$
for positive braids:





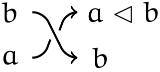
$\mathrm{End}(S^n) \leftarrow B_n^+$	RIII	$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$

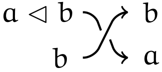


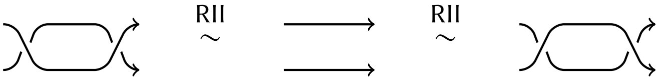
self-distributivity

1 Coloring invariants for braids

Diagram colorings by $(S, \triangleleft)$
for braids:







$\mathrm{End}(S^n) \leftarrow B_n^+$	RIII	$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$	shelf
$\mathrm{Aut}(S^n) \leftarrow B_n$	& RII	$\forall b,\ a \mapsto a \triangleleft b$ invertible	rack
$S \hookrightarrow (S^n)^{B_n}$	(RI)	$a \triangleleft a = a$	quandle

$a \mapsto (a, \ldots, a)$

1 Coloring invariants for braids

$\mathrm{End}(S^n) \leftarrow B_n^+$	RIII	$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$	shelf
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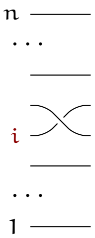
$a \mapsto (a, \ldots, a)$

Examples:

S	$a \triangleleft b$	$(S, \triangleleft)$ is a	in braid theory
${}_{\mathbb{Z}[t^{\pm 1}]}\mathrm{Mod}$	$ta + (1-t)b$	quandle	(red.) Burau: $B_n \to GL_n(\mathbb{Z}[t^{\pm}])$

$\rho_B($  $) = I_{i-1} \oplus \begin{pmatrix} 1-t & 1 \\ t & 0 \end{pmatrix} \oplus I_{n-i-1}$

1 Coloring invariants for braids

$\mathrm{End}(S^n) \leftarrow B_n^+$	RIII	$(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$	shelf
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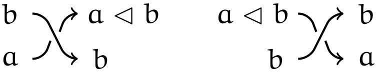
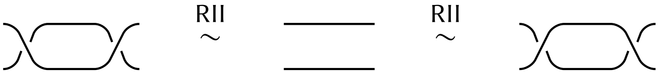
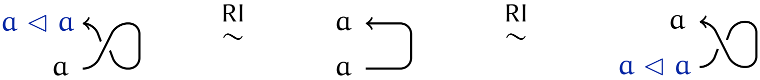
${}_{\mathbb{Z}[t^{\pm 1}]}\mathrm{Mod}$	$ta + (1-t)b$	quandle	(red.) Burau: $B_n \to GL_n(\mathbb{Z}[t^{\pm}])$
group	$b^{-1}ab$	quandle	Artin: $B_n \hookrightarrow \mathrm{Aut}(F_n)$
twisted linear quandle			Lawrence–Krammer–Bigelow
$\mathbb{Z}$	$a+1$	rack	$\lg(w), \mathrm{lk}_{i,j}$
free shelf			Dehornoy: order on B_n

2 Coloring counting invariants for knots

Diagram colorings by $(S, \triangleleft)$
for knots:







pos. braids	RIII	$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$	shelf
braids	& RII	$\forall b,\ a \mapsto a \triangleleft b$ invertible	rack
knots & links	& RI	$a \triangleleft a = a$	quandle

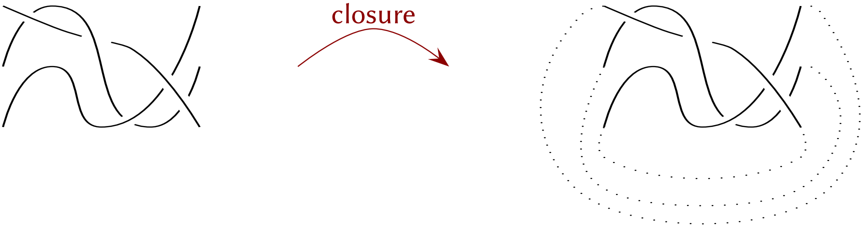
2 Coloring counting invariants for knots

Theorem (*Joyce & Matveev* ’82):

✓ The number of colorings of a diagram D of a knot K by a quandle $(S, \triangleleft)$ yields a knot invariant.

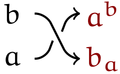
✓ $\#\,\mathrm{Col}_S(D) = \#\,\mathrm{Hom}(Q(K), S) = \mathrm{Tr}(\rho_S(\beta))$

- $Q(K)$ = fundamental quandle of K
  (a weak universal knot invariant);
- $\mathrm{closure}(\beta) = K$;
- $\rho_S \colon B_n \to \mathrm{Aut}(S^n)$ is the S-coloring invariant for braids.



3 Upper strands matter?

Diagram colorings by (S, σ):



$\sigma(a, b) = (b_a, a^b)$

Ex.: $\sigma_{\lhd}(a, b) = (b, a \lhd b)$

RIII-compatibility $\iff$ set-theoretic Yang-Baxter equation:

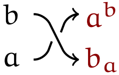
$$\boxed{\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \colon S^{\times 3} \to S^{\times 3}} \qquad \sigma_1 = \sigma \times \mathrm{Id}_S,\ \sigma_2 = \mathrm{Id}_S \times \sigma$$

Set-theoretic solutions $\overset{\text{linearize}}{\rightsquigarrow}$ $\overset{\text{deform}}{\rightsquigarrow}$ linear solutions.

Example: $\sigma(a, b) = (b, a)$ $\rightsquigarrow$ R-matrices.

3 Upper strands matter?

Diagram colorings by (S, σ):



$\sigma(a, b) = (b_a, a^b)$

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RIII-compatibility $\Longleftrightarrow$ set-theoretic Yang-Baxter equation:

$$\boxed{\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 : S^{\times 3} \to S^{\times 3}} \qquad \sigma_1 = \sigma \times \mathrm{Id}_S,\ \sigma_2 = \mathrm{Id}_S \times \sigma$$

Exotic example: $\sigma(a, b) = (b, a)$ $\rightsquigarrow$

$\sigma_{Lie}(a \otimes b) = b \otimes a + \hbar 1 \otimes [a, b]$, where $[1, a] = [a, 1] = 0$:

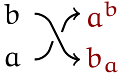
YBE for σ_{Lie} $\Longleftrightarrow$ Leibniz relation for $[\,]$

Very exotic example: $\sigma_{Ass}(a, b) = (a * b, 1)$, where $1 * a = a$:

YBE for σ_{Ass} $\Longleftrightarrow$ associativity for $*$

3 Upper strands matter?

Diagram colorings by (S, σ):



$\sigma(a, b) = (b_a, a^b)$

Ex.: $\sigma_{\lhd}(a, b) = (b, a \lhd b)$

RIII	$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$	YB operator
& RII	σ invertible & $\forall b,\ a \mapsto a^b$ and $a \mapsto a_b$ invertible	birack
& RI	$\exists$ a permutation t of S such that $\sigma(t(a), a) = (t(a), a)$	biquandle

Result: Coloring invariants of braids and knots.

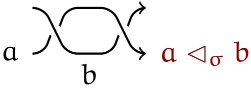
Bad news: These invariants give nothing new!

Unrelated question: Describe free biracks and biquandles.

4 From biracks to racks

Thm (*Soloviev & Lu–Yan–Zhu* ’00, *L.–Vendramin* ’17):

✓ Birack (S, σ) $\rightsquigarrow$ its structure rack $(S, \lhd_\sigma)$:



✓ This is a projection **Birack** $\twoheadrightarrow$ **Rack** along involutive biracks:

- $\lhd_{\sigma_\lhd} = \lhd$;
- $\lhd_\sigma$ trivial $\iff$ $\sigma^2 = \mathrm{Id}$.

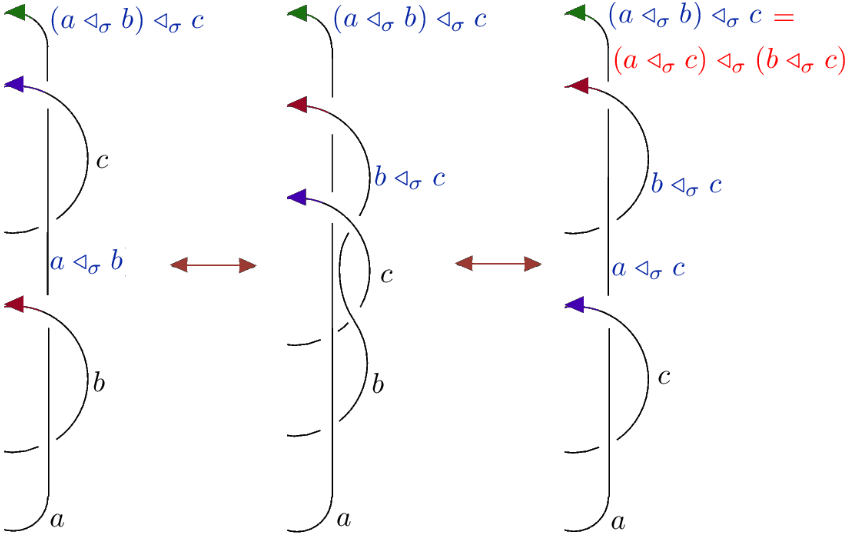
✓ The structure rack remembers a lot about the birack:

- $(S, \lhd_\sigma)$ quandle $\iff$ (S, σ) biquandle;
- σ and $\lhd_\sigma$ induce isomorphic B_n-actions on S^n

$\Longrightarrow$ same braid and knot invariants.

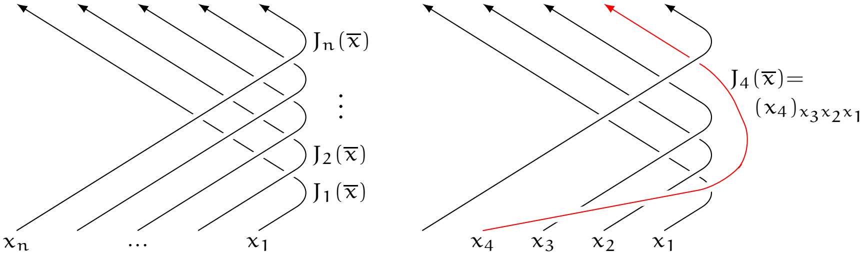
5 Leaping loops

A proof of the self-distributivity of $\lhd_\sigma$:



6 Guitar map

$$\textcolor{green}{J}\colon S^{\times n} \xrightarrow{1:1} S^{\times n},$$
$$(x_n, \ldots, x_1) \longmapsto (\ldots, (x_3)_{x_2 x_1}, (x_2)_{x_1}, x_1).$$



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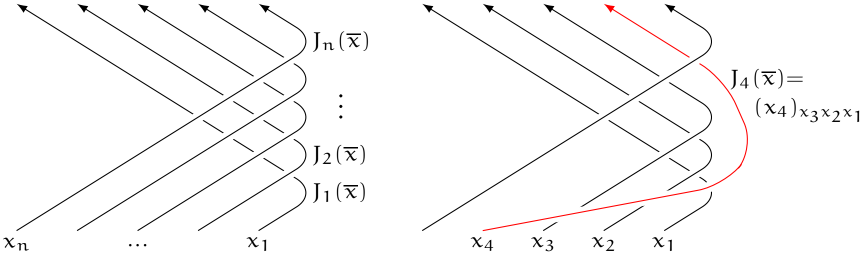


Ex.: $\sigma_{Ass}(a, b) = (ab, 1) \quad \rightsquigarrow \quad J(a, b, c) = (a, ab, abc).$

Ex.: $\sigma_{SD}(a, b) = (b \lhd a, a) \quad \rightsquigarrow \quad J(a, b, c) = (a, b \lhd a, (c \lhd b) \lhd a).$

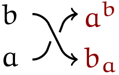
Ex.: $\sigma^2 = \mathrm{Id} \quad \rightsquigarrow \quad \Omega$ from right-cyclic calculus.

6 Guitar map

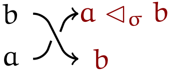
$$J\colon S^{\times n} \xrightarrow{1:1} S^{\times n},$$
$$(x_n, \ldots, x_1) \longmapsto (\ldots, (x_3)_{x_2 x_1}, (x_2)_{x_1}, x_1).$$

Proposition: $\boxed{J\sigma_i = \sigma'_i J}$.

σ:



σ':

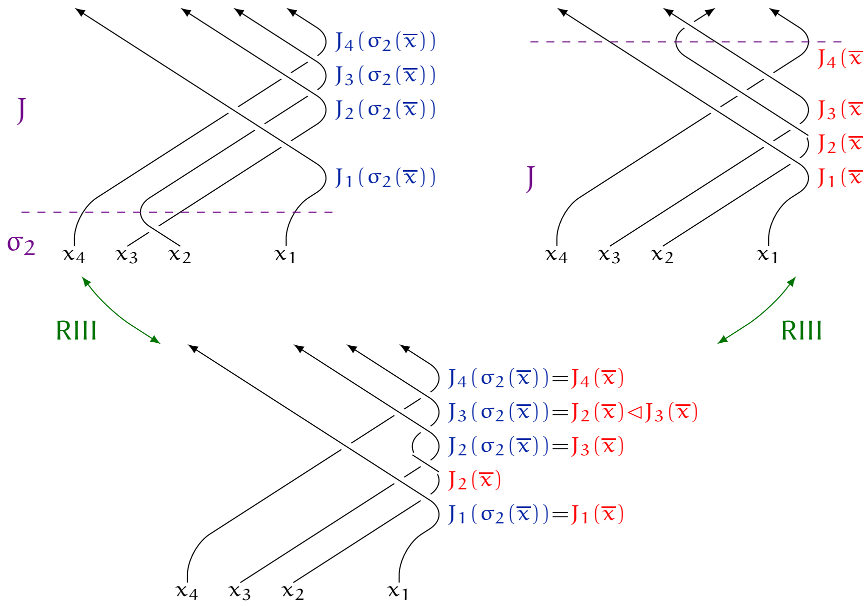


Corollary: Same B_n-actions and knot invariants.

⚠ $(S, \sigma) \not\cong (S, \sigma')$ as biracks!

A recent application (*Blanchet–Geer–Patureau-Mirand–Reshetikhin* '18): holonomy braidings and reps of the unrestricted $U_q\mathfrak{sl}(2)$ at roots of unity.

Proposition: $\boxed{J\sigma_i = \sigma_i' J}$. **Proof**:



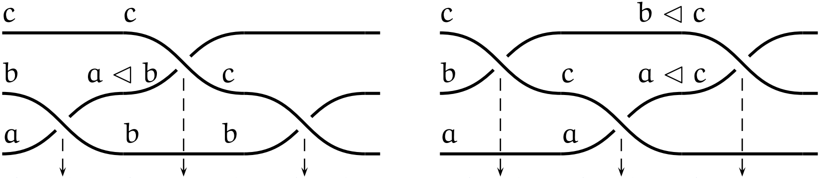
7 Enhancing invariants: weights

Fenn–Rourke–Sanderson ’95 & *Carter–Jelsovsky–Kamada–Langford–Saito* ’03:

Shelf S, $\phi\colon S \times S \to \mathbb{Z}_n$ $\quad\rightsquigarrow\quad$ ϕ-weights:

$$S\text{-colored diagram } D \quad \longmapsto \quad \sum \pm\phi(a, b)$$

The multi-set of weights yields a braid invariant iff



$$\phi(a, b) + \phi(a \lhd b, c) + \cancel{\phi(b, c)} = \cancel{\phi(b, c)} + \phi(a, c) + \phi(a \lhd c, b \lhd c)$$

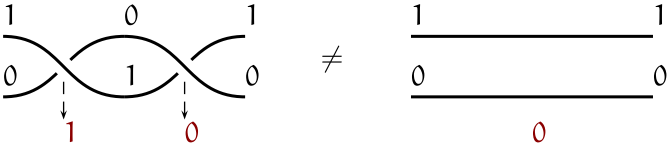
and a knot invariant if moreover $\phi(a, a) = 0$.

7 Enhancing invariants: weights

These ϕ-weights strengthen coloring invariants.

Example: $S = \{0, 1\}$, $a \lhd b = a$,

$\phi(0, 1) = 1$ and $\phi(a, b) = 0$ elsewhere.



Conjecture (*Clark–Saito–…*):

Finite quandle cocycle invariants distinguish all knots.

More generally, this approach works for knottings $K^{n-1} \hookrightarrow \mathbb{R}^{n+1}$.

8 Self-distributive cohomology

$$C_R^k(S, \mathbb{Z}_n) = \mathrm{Map}(S^{\times k}, \mathbb{Z}_n),$$

$$(d_R^k f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} (f(a_1, \ldots, \widehat{a_i}, \ldots, a_{k+1})$$
$$- f(a_1 \lhd a_i, \ldots, a_{i-1} \lhd a_i, a_{i+1}, \ldots, a_{k+1}))$$

$\rightsquigarrow$ Rack cohomology $H_R^k(S, \mathbb{Z}_n)$.

Applications:

① (Higher) braid and knot invariants:

$d_R^2 \phi = 0 \implies \phi$ refines (positive) braid coloring invariants,
$\phi = d_R^1 \psi \implies$ the refinement is trivial.

② Hopf algebra classification (*Andruskiewitsch–Graña* '03).

③ Rack/quandle extensions, deformations etc.

9 Braided cohomology

Carter–Elhamdadi–Saito '04 & *L.* '13:

$$C^k_{\text{Br}}(S, \mathbb{Z}_n) = \text{Map}(S^{\times k}, \mathbb{Z}_n),$$

$$(d^k_{\text{Br}} f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} \big(f(a_1, \ldots, a_{i-1}, (a_{i+1}, \ldots, a_{k+1})_{a_i}) - f((a_1, \ldots, a_{i-1})^{a_i}, a_{i+1}, \ldots, a_{k+1}) \big)$$

$\rightsquigarrow$ Braided cohomology $H^k_{\text{Br}}(S, \mathbb{Z}_n)$.

Applications:

① (Higher) braid and knot invariants:

$d^2_{\text{Br}}\phi = 0 \implies \phi$ refines (positive) braid coloring invariants,
$\phi = d^1_{\text{Br}}\psi \implies$ the refinement is trivial.

Question: New invariants?

Answer: I don't know!

Braided cohomology

Applications:

② $d^2_{Br}\phi = 0 \implies$ diagonal deformations of σ:

$$\sigma_q(a,b) = q^{\phi(a,b)}\sigma(a,b).$$

(*Freyd–Yetter* '89, *Eisermann* '05)

③ Unifies cohomology theories for

✓ self-distributive structures $\sigma_{SD}(a,b) = (b \lhd a, a)$

✓ associative structures $\sigma_{Ass}(a,b) = (a * b, 1)$

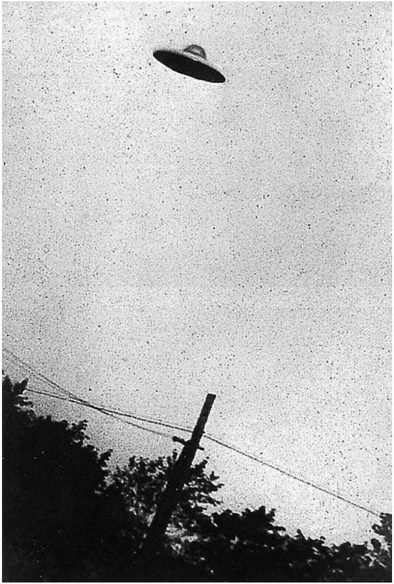
✓ Lie algebras $\sigma_{Lie}(a \otimes b) = b \otimes a + \hbar 1 \otimes [a,b]$

....

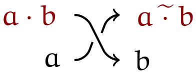
+ explains parallels between them
+ suggests theories for new structures.

④ Computes the cohomology of certain monoids.

10 Flying saucer cohomology



10 Flying saucer cohomology

Sideways maps: 

Fenn–Rourke–Sanderson '93, *Ceniceros–Elhamdadi–Green–Nelson* '14:

$$C^k_{\text{Bir}}(S, \mathbb{Z}_n) = \mathrm{Map}(S^{\times k}, \mathbb{Z}_n),$$

$$(d^k_{\text{Bir}} f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} \big(f(a_1, \ldots, \widehat{a_i}, \ldots, a_{k+1}) - f(a_i \,\tilde{\cdot}\, a_1, \ldots, a_i \,\tilde{\cdot}\, a_{i-1}, a_i \cdot a_{i+1}, \ldots, a_i \cdot a_{k+1}) \big)$$

$\rightsquigarrow$ Birack cohomology $H^k_{\text{Bir}}(S, \mathbb{Z}_n)$.

Normalized subcomplex C^k_N for biquandles: $f(\ldots, a_i, a_i, \ldots) = 0$.

Application: Braid and knot invariants.

10 Flying saucer cohomology

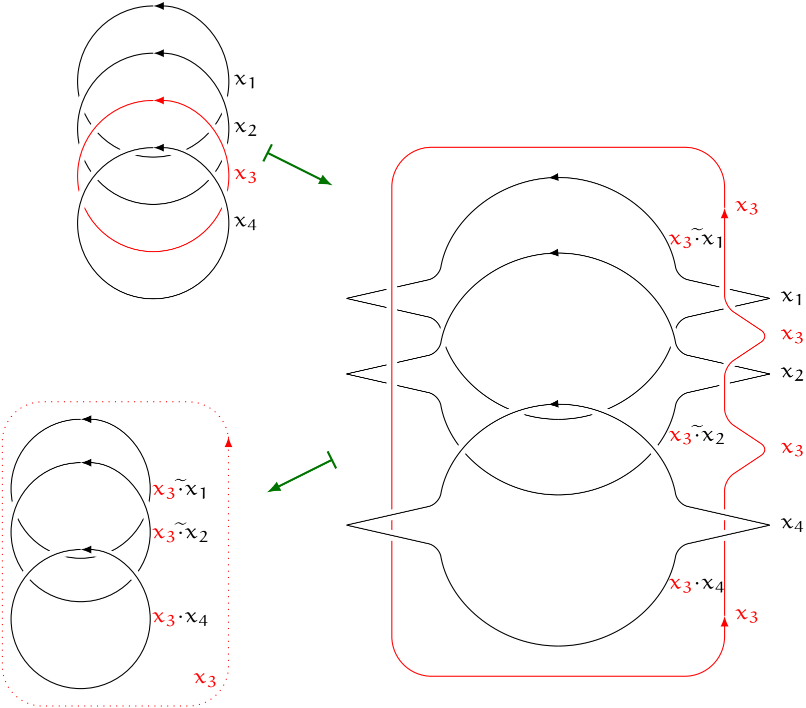
Thm (*L.–Vendramin* '17):

✓ Braided and birack cohomologies are the same:

$$J^* \colon (C^\bullet_{\text{Bir}}(S, \mathbb{Z}_n), d^\bullet_{\text{Bir}}) \cong (C^\bullet_{\text{Br}}(S, \mathbb{Z}_n), d^\bullet_{\text{Br}}).$$

✓ For biquandles, cohomology decomposes: $C^\bullet_{\text{Bir}} \cong C^\bullet_{N} \oplus C^\bullet_{D}$.

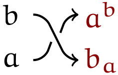
Proof: Guitar map counter-attacks, or Loops leap again!

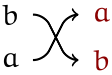


x_1
x_2
x_3
x_4
x_3
$x_3 \tilde{\cdot} x_1$
x_1
x_3
x_2
$x_3 \tilde{\cdot} x_2$
x_3
x_4
$x_3 \cdot x_4$
x_3
$x_3 \tilde{\cdot} x_1$
$x_3 \tilde{\cdot} x_2$
$x_3 \cdot x_4$
x_3

11 Getting virtual

Virtual diagram colorings by (S, σ):





$\sigma(a, b) = (b_a, a^b)$

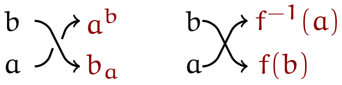
Result:

- ✓ coloring invariants for virtual braids/knots;
- ✓ $\sigma_{\triangleleft}(a, b) = (b, a \triangleleft b) \rightsquigarrow$ coloring invariants for welded braids/knots;
- ✓ weight enhancements available.

11 Getting virtual

Manturov ’02:

Virtual diagram colorings by (S, σ, f):



$\sigma(a, b) = (b_a, a^b)$

$\sigma(f(a), f(b)) = (f(b_a), f(a^b))$

Example: S is a rack, $c \in S$, $f(a) = a \triangleleft c$.

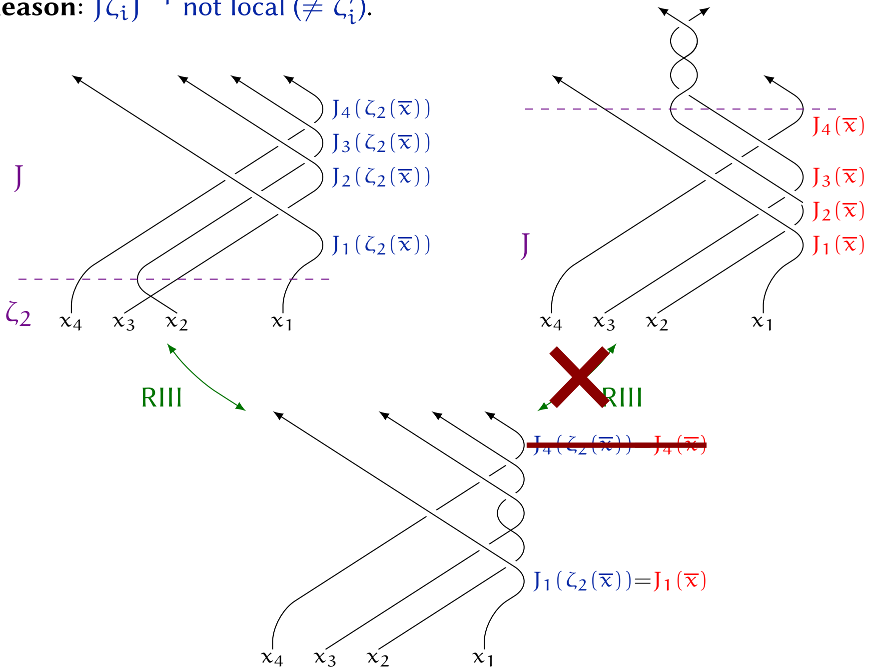
Result: coloring invariants for virtual braids/knots.

12 The right setting for biracks?

It seems so:

For virtual/welded braids, birack coloring invariants seem richer than rack coloring invariants.

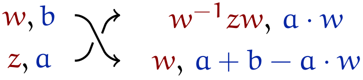
Reason: $J\zeta_i J^{-1}$ not local ($\neq \zeta_i'$).

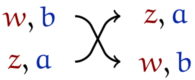


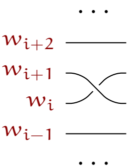
12 The right setting for biracks?

Example: the welded Leeds biquandle (*Kauffman–Faria Martins* '08, *Bullivant–Faria Martins–Martin* '18):

$S = G \times A$, where G is a group acting on an abelian group A,





$\rho($  $) = I_{i-1} \oplus \begin{pmatrix} 1 - w_{i+1} & w_{i+1} \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}$

$\rightsquigarrow$ coloring invariants for welded knots;

no obvious welded quandle yielding the same invariants.

13 Virtual braids seen categorically

Thm: $B_n \simeq \mathrm{End}_{\mathcal{C}_{br}}(V^{\otimes n})$,

$\mathcal{C}_{br}$ = the free braided category generated by an object V.

Crl: object X in a braided category $\rightsquigarrow$ a B_n-rep. on $X^{\otimes n}$.

Thm (*L.* '13, *Brochier* '16): $VB_n \simeq \mathrm{End}_{\mathcal{C}_{2br}}(V^{\otimes n})$,

$(\mathcal{C}_{2br}, c)$ = the free symmetric category gen. by a braided object (V, σ).

category level	global symmetry c	local braiding σ for V
VB_n level	S_n part	B_n part

Crl: braided object X in a symmetric category $\rightsquigarrow$ a VB_n-rep. on $X^{\otimes n}$.

Example: Quantum invariants of knots extend to virtual knots.

Question: Categorical interpretation for WB_n?