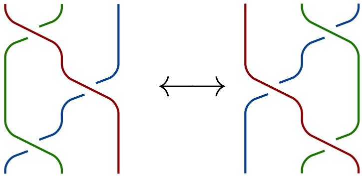
What do braids know about Young tableaux?

Victoria LEBED

Trinity College Dublin

Marburg, February 2017

3		
2	6	6
1	4	5



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✓ Crystal bases for quantum groups (90').

2 Schensted algorithm

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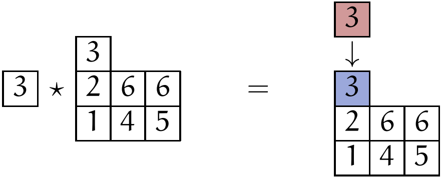

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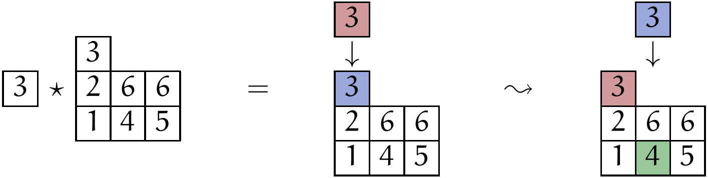

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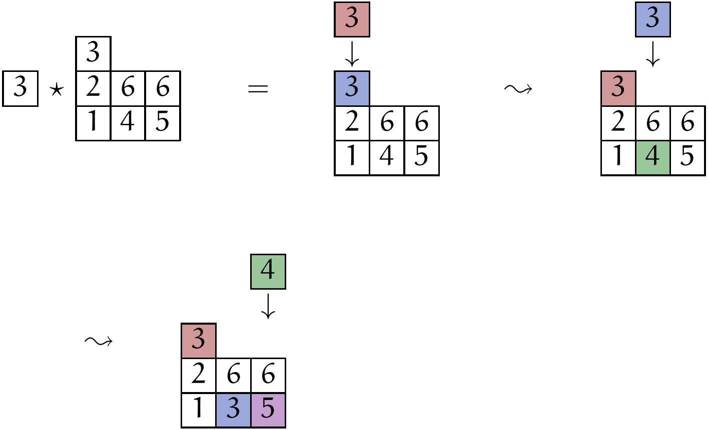

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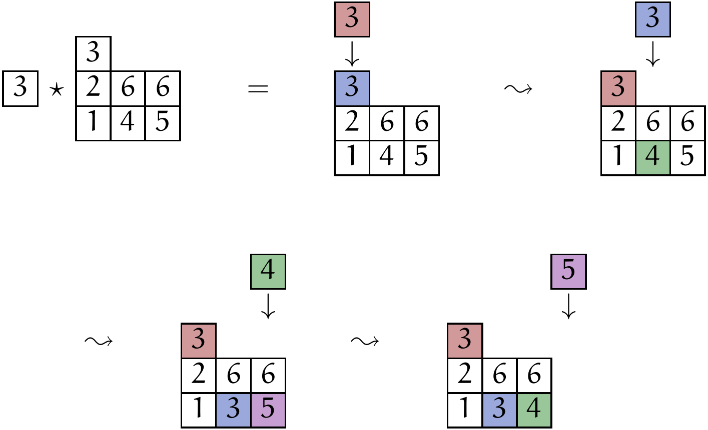

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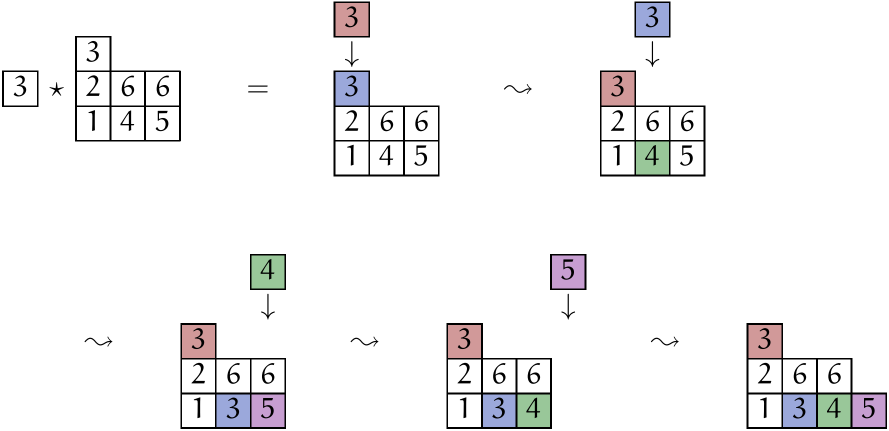

3 Plactic monoid

Tableaux vs. words: $A_n = \{1, 2, \ldots, n\}$,

$$\mathcal{C}, \mathcal{R} \quad : \quad \mathbf{YT}_n \quad \rightleftarrows \quad A_n^* \quad : \quad \mathcal{T}$$

$$T = \begin{array}{|c|c|c|}\hline 3 & & \\ \hline 2 & 6 & 6 \\ \hline 1 & 4 & 5 \\ \hline \end{array} \quad \overset{\mathcal{R}}{\longmapsto} \quad 3\,266\,145$$

$$\overset{\mathcal{C}}{\longmapsto} \quad 321\;64\;65$$

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Thm (*Knuth* '70): $(\mathbf{YT}_n, \star) \overset{\text{iso}}{\longleftrightarrow} (\, \mathcal{A}_n^*/\!\sim, \mathsf{concat}\,)$

$$xzy \sim zxy, \quad x \leqslant y < z;$$

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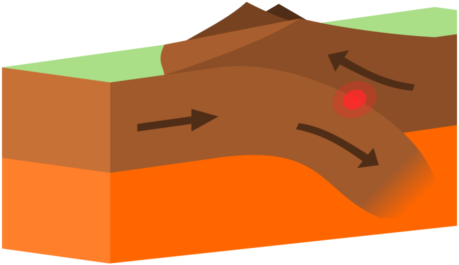

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Example: $\mathbf{Col}_2^{\bullet} = \left\{ \boxed{1}, \boxed{2}, \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array} \right\}$,

$$\boxed{2} \cdot \boxed{1} \to \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array}, \qquad \boxed{i} \cdot \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array} \to \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array} \cdot \boxed{i}.$$

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Applications:

- rewriting;
- cohomological computations (cf. *Lopatkin* '16).

6 Yang–Baxter equation

Data:

- monoidal category $\mathcal{C}$ (= $\mathbf{Vect}_{\Bbbk}$);
- object S;
- morphism $\sigma\colon S \otimes S \to S \otimes S$.

YBE: $\boxed{\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \colon S^{\otimes 3} \to S^{\otimes 3}}$ $\qquad \sigma_1 = \sigma \otimes \mathrm{Id}_S,\ \sigma_2 = \mathrm{Id}_S \otimes \sigma$

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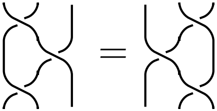
Topological avatar:

$\sigma \longleftrightarrow$ 

YBE $\longleftrightarrow$ 

Reidemeister III move

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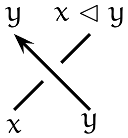

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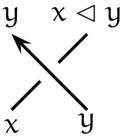

- Hopf algebra classification (*Andruskiewitsch–Graña* '03).

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Example: free self-distributive structures.

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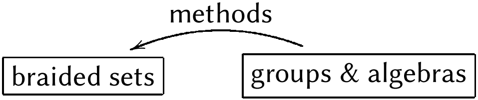
✓ Plactic monoid: $\sigma_{n,c}$ on $\mathbf{Col}_n$.

10 Universal enveloping constructions

Universal enveloping monoids:

$$\mathrm{Mon}(S,\sigma) = \langle\, S \mid xy = y'x' \text{ whenever } \sigma(x,y) = (y',x') \,\rangle$$

U. e. groups and algebras are defined similarly.



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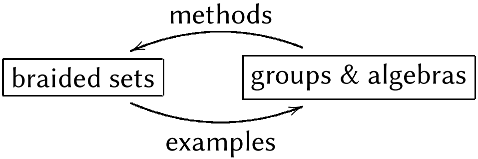

Theorem: (S,σ) a "nice" finite braided set, $\sigma^2 = \mathrm{Id} \implies$

✓ $\mathrm{Mon}(S,\sigma)$ is of I-type, cancellative, Ore;

✓ $\mathrm{Grp}(S,\sigma)$ is solvable, Garside;

✓ $\Bbbk\,\mathrm{Mon}(S,\sigma)$ is Koszul, noetherian, Cohen–Macaulay, Artin–Schelter regular

(*Manin, Gateva-Ivanova & Van den Bergh, Etingof–Schedler–Soloviev, Jespers–Okniński, Chouraqui 80'-…*).

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Universal enveloping monoids:

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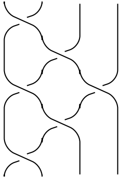
$\Delta_4 =$ 

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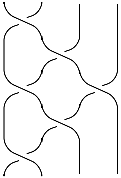
$\Delta_4 =$ 

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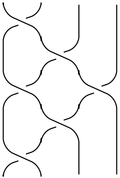
$\Delta_4 =$ 

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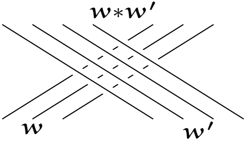

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Example: $(\mathbf{Norm}(\mathbf{Col}_n, \sigma_{n,c}, e_c), *_{br}) \simeq (\mathbf{YT}_n, \star_{Sch})$.

A cohomology theory?

A cohomology theory for braided sets should:

1) Describe diagonal deformations (*Freyd–Yetter* '89, *Eisermann* '05):

$$\sigma_q(x,y) = q^{\omega(x,y)}\sigma(x,y),\ \omega\colon S \times S \to \mathbb{Z}_m.$$

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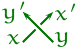
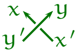
2) Yield knot and knotted surface invariants (*Carter et al.* '01):

(S,σ)-coloured diagram $(D,\mathcal{C})$ & $\omega\colon S\times S\to\mathbb{Z}$

$\rightsquigarrow$ Boltzmann weight $\mathscr{B}_\omega(\mathcal{C}) = \sum \omega(x,y) - \sum \omega(x,y).$





ω a 2-cocycle $\Longrightarrow$ a knot invariant given by

$$\{\,\mathscr{B}_\omega(\mathcal{C})\mid \mathcal{C}\text{ is a }(S,\sigma)\text{-colouring of }D\,\}.$$

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A cohomology theory for braided sets should:

3) Unify cohomology theories for

- ✦ associative structures,
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4) Compute the cohomology of $\Bbbk\,\mathrm{Mon}(S,\sigma)$.

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Data:

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$$d^n = \sum_{i=1}^{n+1} (-1)^{i-1} (d_l^{n;i} - d_r^{n;i}),$$

$$\begin{array}{c} \varepsilon(x'_i) f(x'_1 \ldots x'_{i-1} x_{i+1} \ldots x_{n+1}) \\ \uparrow \\ x'_i x'_1 \ldots x'_{i-1} x_{i+1} \ldots x_{n+1} \\ {\scriptstyle \sigma_1 \cdots \sigma_{i-1}} \uparrow \\ x_1 \ldots x_{n+1} \end{array}$$

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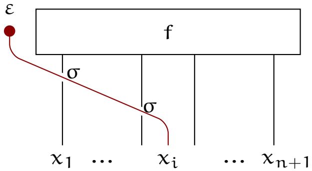

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Versions:

- diagrammatic;
- algebraic: quantum shuffles (*Rosso* '95).

A good theory?

1) & 2) For $\omega \in C^2(S, \sigma; \mathbb{Z}_m, \varepsilon_1)$,

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3) Unifies classical cohomology theories.

Comparing cohomologies

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braided cohomology		Hochschild cohomology
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Application: Spectral sequence for factorised monoids $G = HK$.

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$$HH^1 = \Bbbk f_{\boxed{1}} \oplus \Bbbk f_{\boxed{2}}, \qquad HH^3 = \Bbbk f_{\boxed{2}, \boxed{1}, \begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}};$$

$$HH^2 = \Bbbk f_{\boxed{1}, \begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}} \oplus \Bbbk f_{\boxed{2}, \begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}}, \qquad HH^k = 0, \; k > 3.$$

$$f_{\boxed{2}} \smile f_{\boxed{1}, \begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}} = -f_{\boxed{2}, \begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}} \smile f_{\boxed{1}} = f_{\boxed{2}, \boxed{1}, \begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}}.$$

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For the braided character $\varepsilon_0 \colon c \mapsto 0, e_c \mapsto 1$,

Thm:
1) $HH^1(\Bbbk\mathbf{Pl}_n; \Bbbk, \varepsilon_0) \simeq A_n^\vee \qquad (:= \mathrm{Maps}(A_n, \Bbbk))$.
2) $HH^2 \simeq \left(\begin{smallmatrix}\square & \square \\ & \square\end{smallmatrix}\right)^\vee$.
3) $HH^1 \smile HH^1 = 0$.
4) $HH^j \neq 0$ for all j when $n > 2$.
5) For $n = 2$,

$$HH^1 = \Bbbk f_{\boxed{1}} \oplus \Bbbk f_{\boxed{2}}, \qquad HH^3 = \Bbbk f_{\boxed{2},\boxed{1},\substack{\boxed{2}\\\boxed{1}}};$$

$$HH^2 = \Bbbk f_{\boxed{1},\substack{\boxed{2}\\\boxed{1}}} \oplus \Bbbk f_{\boxed{2},\substack{\boxed{2}\\\boxed{1}}}, \qquad HH^k = 0,\ k > 3.$$

$$f_{\boxed{2}} \smile f_{\boxed{1},\substack{\boxed{2}\\\boxed{1}}} = -f_{\boxed{2},\substack{\boxed{2}\\\boxed{1}}} \smile f_{\boxed{1}} = f_{\boxed{2},\boxed{1},\substack{\boxed{2}\\\boxed{1}}}.$$

Crl: cohomological dimension:

$\mathrm{cd}(\mathbf{Pl}_n) =$	∞,	3,	1
for n	> 2,	$= 2$,	$= 1$.