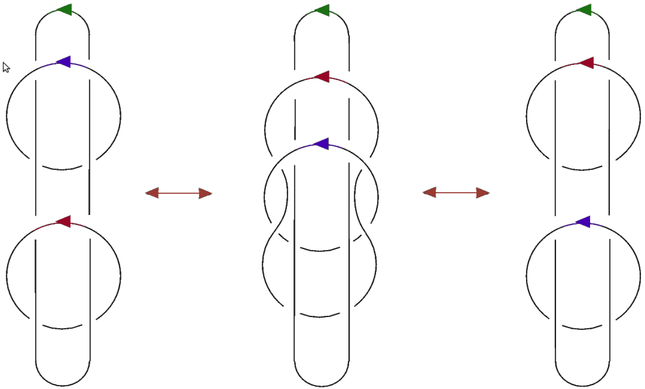
On set-theoretic solutions to the Yang-Baxter equation

Victoria LEBED (Nantes)
with Leandro VENDRAMIN (Buenos Aires)



Mulhouse, October 21, 2015

1 Yang-Baxter equation

✓ V: vector space,
✓ $\sigma\colon V^{\otimes 2} \to V^{\otimes 2}$.

Yang-Baxter equation (YBE)

$$\sigma_1 \circ \sigma_2 \circ \sigma_1 = \sigma_2 \circ \sigma_1 \circ \sigma_2 \colon V^{\otimes 3} \to V^{\otimes 3}$$

where $\sigma_1 = \sigma \otimes \mathrm{Id}_V$, $\sigma_2 = \mathrm{Id}_V \otimes \sigma$.

Origins:
→ factorization condition for the dispersion matrix in the 1-dim. n-body problem (*McGuire, Yang, 60'*);
→ partition function for exactly solvable lattice models (*Baxter, 70'*).

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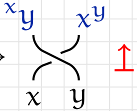
$\sigma \longleftrightarrow$ 

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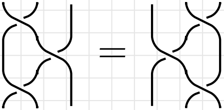

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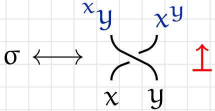
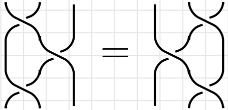
Left non-degenerate braided set: for all y, $x \mapsto x^y$ is a bijection $X \to X$.





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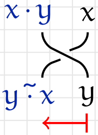

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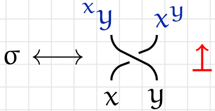

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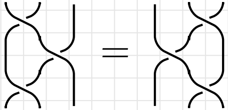

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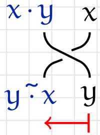

Birack: left and right non-degenerate braided set with invertible σ.

2 Structure (semi)group

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Structure (semi)group of (V,σ): $(S)G_{X,\sigma} = \langle\, X \mid xy = {}^{x}y x^{y} \,\rangle$

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→ A source of interesting groups and algebras:
Theorem: If (X,σ) is a finite RI-compatible birack with $\sigma^2 = \mathrm{Id}$, then

- ✓ $SG_{X,\sigma}$ is of I-type, cancellative, Öre;
- ✓ $G_{X,\sigma}$ is solvable, Garside;
- ✓ $\Bbbk SG_{X,\sigma}$ is Koszul, noetherian, Cohen-Macaulay, Artin-Schelter regular

(*Manin, Gateva-Ivanova & Van den Bergh, Etingof-Schedler-Soloviev, Jespers-Okniński, Chouraqui, 80'-. . .*).

3 Self-distributive structures

Shelf: set X & $\lhd\colon X \times X \to X$ s.t.

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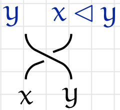
$\Leftrightarrow \sigma_{\lhd} =$ 

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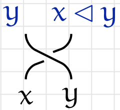
$\Leftrightarrow \sigma_{\lhd} =$ 

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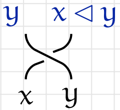
$\Leftrightarrow$ $\sigma_{\triangleleft} =$ 

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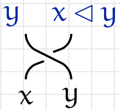
$\Leftrightarrow$ $\sigma_{\triangleleft} =$ 

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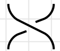
$\Leftrightarrow$ $\sigma_{\triangleleft} =$ 

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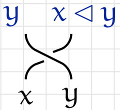
$\Leftrightarrow \sigma_\lhd =$ 

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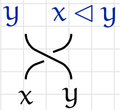
$\Leftrightarrow$ $\sigma_{\triangleleft}=$ 

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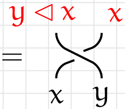
$\Leftrightarrow$ $\sigma_\triangleleft{}'=$ 

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Cycle set: set X & $\cdot : X \times X \to X$ s.t.

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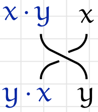

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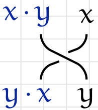

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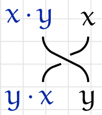
$\Leftrightarrow \sigma_{\cdot} =$ 

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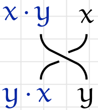
$\Leftrightarrow \sigma_{\cdot} =$ 

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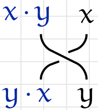

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For a monoid $(X, \star, 1)$,
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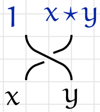
$\Leftrightarrow \sigma_\star =$ 

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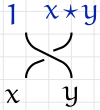

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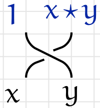

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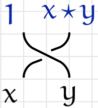

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One has a semigroup isomorphism

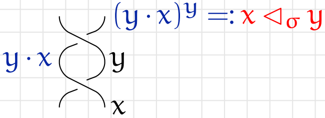
$$SG_{X,\sigma_\star} \xrightarrow{\sim} X,$$

$$x_1 \cdots x_k \mapsto x_1 \star \cdots \star x_k.$$

6 Associated shelf

Fix a LND braided set (X, σ).

Proposition (*L.-V. 2015*): one has a shelf $(X, \lhd_\sigma)$, where



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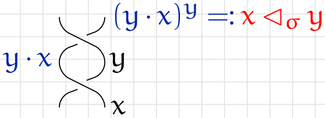
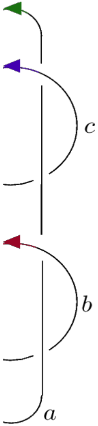


<u>Proof</u>:



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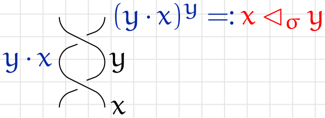

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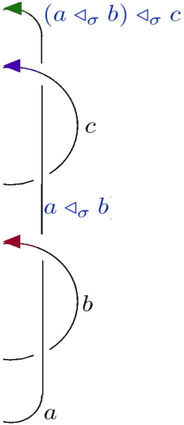

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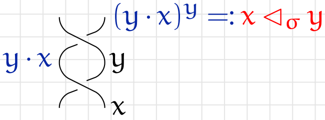

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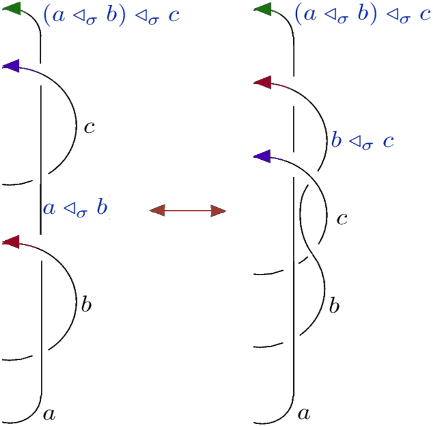

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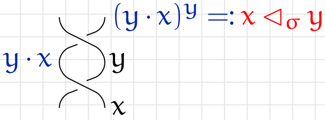

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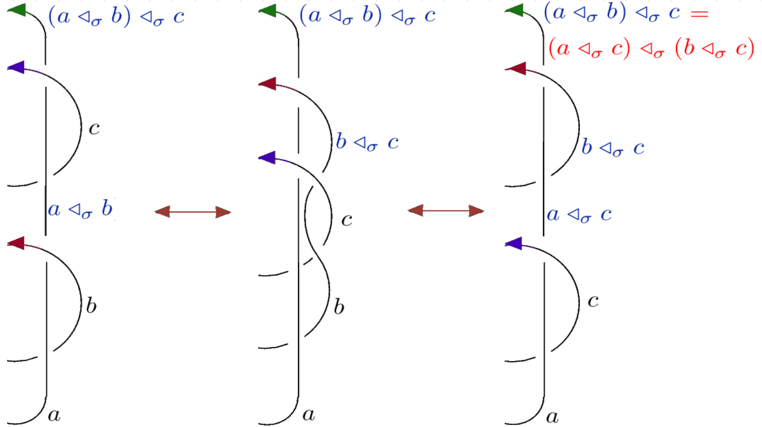

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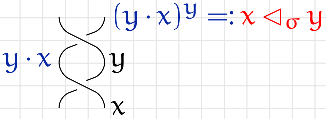

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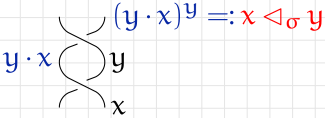

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Proposition (*L.-V. 2015*):

- → $(X, \triangleleft_\sigma)$ is a rack $\Leftrightarrow$ σ is invertible;
- → $(X, \triangleleft_\sigma)$ is a trivial $(x \triangleleft_\sigma y = x)$ $\Leftrightarrow$ $\sigma^2 = \mathrm{Id}$;
- → $x \triangleleft_\sigma x = x$ $\Leftrightarrow$ $\sigma(x \cdot x, x) = (x \cdot x, x)$.

7 Guitar map

$$J^{(n)} \colon X^{\times n} \xrightarrow{\sim} X^{\times n},$$

$$(x_1, \ldots, x_n) \mapsto (x_1^{x_2 \cdots x_n}, \ldots, x_{n-1}^{x_n}, x_n),$$

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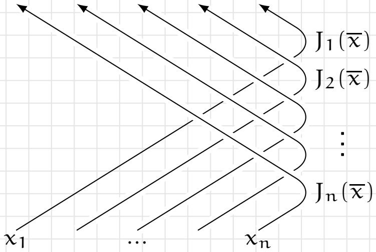

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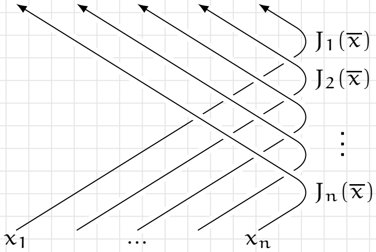

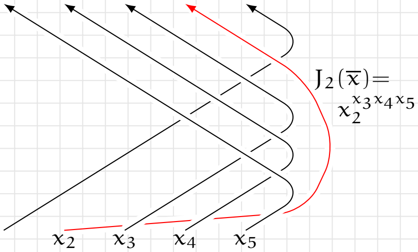

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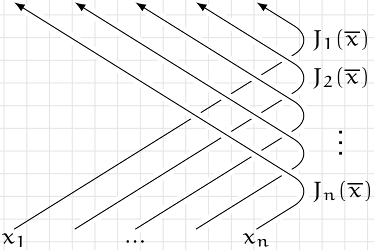

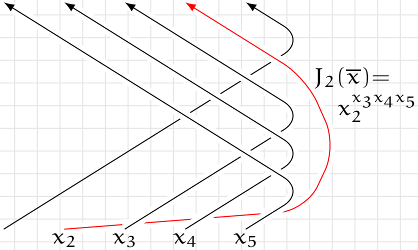

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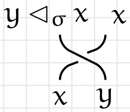

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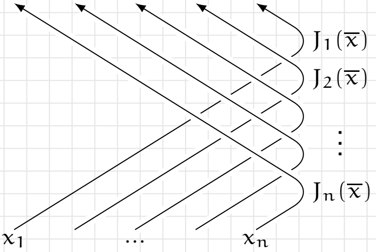
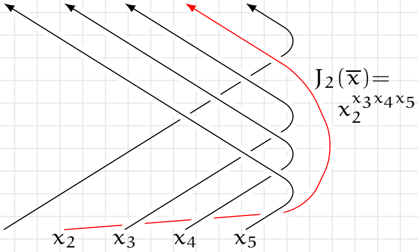




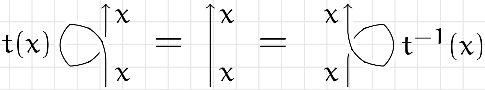
Proposition (*L.-V. 2015*): $\boxed{J\sigma_i = \sigma'_i J}$, where $\sigma' = \sigma'_{\triangleleft_\sigma}$.

Corollary: σ and σ' yield isomorphic B_n-actions on $X^{\times n}$.

Warning: In general, $(X, \sigma) \not\cong (X, \sigma')$ as braided sets!

RI-compatibility

RI-compatible braiding: $\exists t\colon X \xrightarrow{\sim} X$ s.t. $\sigma(t(x), x) = (t(x), x)$.



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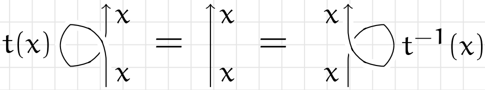

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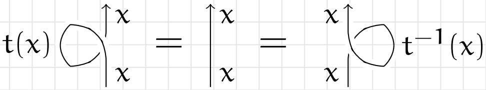

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→ for a rack, it means $x \lhd x = x$ (here $t(x) = x$);

→ for a cycle set, it means the non-degeneracy (here $t(x) = x \cdot x$).

9 Structure group via associated shelf

Theorem (*L.-V. 2015*): (1) The guitar maps induce a bijective 1-cocycle $\boxed{J\colon SG_{X,\sigma} \xrightarrow{\sim} SG_{X,\sigma'}}$, where $\sigma' = \sigma'_{\triangleleft_\sigma}$.

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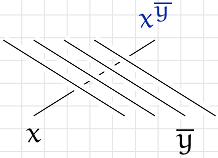

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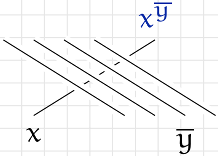

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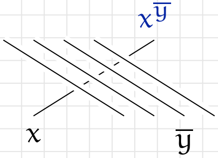

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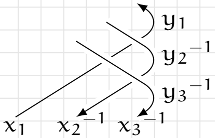

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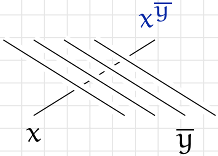

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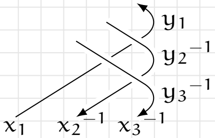

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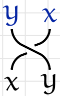
→ $J : \sigma_\cdot \leftrightarrow$ flip 

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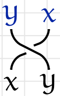
→ $J\colon \sigma_\cdot \leftrightarrow$ flip 

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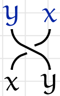
→ $J \colon \sigma_\cdot \leftrightarrow$ flip 

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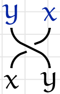
→ $J \colon \sigma_\cdot \leftrightarrow$ flip 

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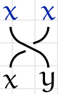
→ $J \colon \sigma_\star \leftrightarrow$ 

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→ $SG_{X,\sigma'_\star} \xrightarrow{\sim} X, \qquad x_1 \cdots x_k \mapsto x_1$.