Yang–Baxter equation, Young tableaux, group factorisations

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2	6	6
1	3	4

$$(ab)c = a(bc)$$

$$[x,[y,z]] = [[x,y],z] - [[x,z],y]$$

GDR Topologie Algébrique et Applications
Colloque 2016

1 Yang–Baxter equation: basics

Data: monoidal category $\mathcal{C}$ (mainly sets / vector spaces),
object V, $\sigma \colon V^{\otimes 2} \to V^{\otimes 2}$.

Yang-Baxter equation (YBE)

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \colon V^{\otimes 3} \to V^{\otimes 3}$$

$$\sigma_1 = \sigma \otimes \mathrm{Id}_V, \ \sigma_2 = \mathrm{Id}_V \otimes \sigma$$

Omnipresent:

- particle physics;
- statistical mechanics;
- quantum / conformal field theory;
- quantum groups;
- C^* algebras;

....

1 Yang–Baxter equation: basics

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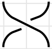
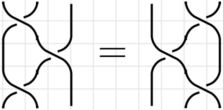
Omni-present:

➻ low-dimensional topology

$\sigma \longleftrightarrow$  ↥

YBE $\longleftrightarrow$  Reidemeister III move

2 Basic examples

Data: monoidal category $\mathcal{C}$, object V, $\sigma\colon V^{\otimes 2} \to V^{\otimes 2}$.

YBE: $\boxed{\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\colon V^{\otimes 3} \to V^{\otimes 3}}$ $\sigma_1 = \sigma \otimes \mathrm{Id}_V, \ \ldots$

✓ $\sigma = \mathrm{Id}_{V \otimes V}$;

✓ $\mathcal{C} = \mathbf{Set}$, $\sigma(x,y) = (y,x)$ $\overset{\text{deform}}{\rightsquigarrow}$ $\mathcal{C} = \mathbf{Vect}_{\Bbbk}$,

- ✓ $\sigma =$ R-matrix (quantum groups);
- ✓ Lie (Leibniz) algebra $(V, [])$, central element $1 \in V$, $\sigma(x \otimes y) = y \otimes x + \hbar 1 \otimes [x,y]$.

YBE for σ $\iff$ Jacobi identity for $[]$

✓ rack $(S, \triangleleft)$, $\sigma(x,y) = (y, x \triangleleft y)$

YBE for σ $\iff$ self-distributivity for $\triangleleft$

Self-distributivity: $\boxed{(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)}$

3 Idempotent YBE solutions

✓ Monoid $(S, *, 1)$, $\sigma(x, y) = (1, x * y)$;

YBE for σ $\iff$ associativity for $*$

✓ Monoid factorisation $G = HK$,

$S = H \cup K$, $\sigma(x, y) = (h, k)$, $h \in H$, $k \in K$, $hk = xy$.

✓ Ordered set S,

$\sigma(x, y) = (x, \max\{x, y\})$.

$\sigma(x, y) = (\min\{x, y\}, \max\{x, y\})$.

✓ Lattice $(S, \wedge, \vee)$, $\sigma(x, y) = (x \wedge y, x \vee y)$.

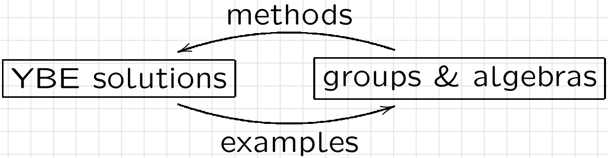
All these YBE solutions are idempotent: $\sigma^2 = \sigma$.

4 Universal enveloping constructions

Un. env. monoid of a YBE solution (S,σ) in $\mathcal{C}=\mathbf{Set}$:

$$\mathrm{Mon}(S,\sigma)=\langle\, S \mid xy=y'x' \text{ whenever } \sigma(x,y)=(y',x')\,\rangle$$

Un. env. (semi)groups and algebras are defined similarly.



Theorem: (S,σ) a "nice" finite YBE solution, $\sigma^2=\mathrm{Id} \Longrightarrow$

- ✓ $\mathrm{Mon}(S,\sigma)$ is of I-type, cancellative, Ore;
- ✓ $\mathrm{Grp}(S,\sigma)$ is solvable, Garside;
- ✓ $\Bbbk\mathrm{Mon}(S,\sigma)$ is Koszul, noetherian, Cohen–Macaulay, Artin–Schelter regular

(*Manin, Gateva-Ivanova & Van den Bergh, Etingof–Schedler–Soloviev, Jespers–Okniński, Chouraqui 80'-. . .*).

4 Universal enveloping constructions

$$\mathrm{Mon}(S,\sigma) = \langle\, S \,|\, xy = y'x' \text{ whenever } \sigma(x,y) = (y',x') \,\rangle$$

Examples:

✓ monoid $(S, *, 1)$, $\quad \sigma(x,y) = (1, x*y)$,

$$S \simeq {}^{\mathrm{Mon}(S,\sigma)}/_{(\star)};$$

✓ Lie algebra $(V, [\,], 1)$, $\quad \sigma(x \otimes y) = y \otimes x + 1 \otimes [x,y]$,

$$UEA(V,[\,]) \simeq {}^{\mathrm{Alg}(V,\sigma)}/_{(\star)}.$$

$(\star)$: $\quad 1 = 1_{\mathrm{Mon}}$

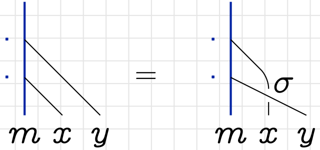
5 Representations

$\mathrm{Mon}(S,\sigma) = \langle\, S \mid xy = y'x' \text{ whenever } \sigma(x,y) = (y',x') \,\rangle$

Representations of (S,σ) := representations of $\Bbbk\,\mathrm{Mon}(S,\sigma)$,

i.e. $\Bbbk$-vector spaces M with $M \times S \xrightarrow{\cdot} M$ s.t.

$$(m \cdot x) \cdot y \quad = \quad (m \cdot y') \cdot x'$$



Examples:

- trivial representation: $M = \Bbbk$, $m \cdot x = m$;
- $M = \Bbbk\,\mathrm{Mon}(S,\sigma)$, $m \cdot x = mx$;
- usual reps for monoids, Lie algebras, racks.

6 A cohomology theory?

A cohomology theory for YBE solutions should:

1) Describe deformations: $\sigma_0 \rightsquigarrow \sigma_0 + \hbar\sigma_1 + \hbar^2\sigma_2 + \cdots$.

Difficult! Pioneers: *Freyd–Yetter '89, Eisermann '05*.

First approximation: diagonal deformations (in some cases yield all deformations)

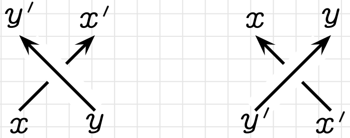
$$\sigma_q(x,y) = q^{\omega(x,y)}\sigma(x,y),\ \omega\colon S \times S \to \mathbb{Z}.$$

ω a 2-cocycle $\Longrightarrow$ σ_q a YBE solution.

6 A cohomology theory?

Theorem: For "nice" set-theoretic YBE solutions (S, σ), one has efficient and easily computable knot and knotted surface invariants

$$\#\{\ (S,\sigma)\text{-colourings of diagrams }\}.$$



$$\sigma(x, y) = (x', y')$$

Example: Solutions coming from racks are "nice".

6 A cohomology theory?

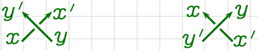
A cohomology theory for YBE solutions should:

2) Enhance the above invariants:

(S,σ)-coloured diagram $(D,\mathcal{C})$ & $\omega\colon S\times S\to\mathbb{Z}$

$\rightsquigarrow$ Boltzmann weight

$$\mathcal{B}_\omega(\mathcal{C}) = \sum \omega(x,y) - \sum \omega(x,y).$$



ω a 2-cocycle $\Longrightarrow$ a knot invariant is defined by $\{\mathcal{B}_\omega(\mathcal{C}) \,|\, \mathcal{C}$ is a (S,σ)-colouring of $D\}$.

6 A cohomology theory?

A cohomology theory for YBE solutions should:

3) Unify cohomology theories for

- associative structures,
- Lie algebras,
- racks etc.

+ explain parallels between them,

+ suggest theories for new structures.

4) Be interpreted in terms of classifying spaces.

5) Compute the cohomology of $\Bbbk\mathrm{Mon}(S, \sigma)$.

7 Braided cohomology

<u>Data</u>: $\mathcal{C} = \mathbf{Set}$, YBE solution (S, σ), bimodule M over it.

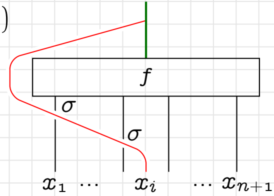
(any preadditive monoidal category will do)

<u>Construction</u>:

$C^n(S, \sigma; M) = \mathrm{Maps}(S^{\times n}, M)$,

$d^n = \sum_{i=1}^{n+1} (-1)^{i-1} (d_l^{n;i} - d_r^{n;i}) \colon C^n \to C^{n+1}$,

$$d_l^{n;i} f: \quad x_1 \ldots x_{n+1} \xrightarrow{\sigma_1 \cdots \sigma_{i-1}} x_i' x_1' \ldots x_{i-1}' x_{i+1} \ldots x_{n+1} \to x_i' \cdot f(x_1' \ldots x_{i-1}' x_{i+1} \ldots x_{n+1})$$



<u>Theorems</u>:

➻ This defines a precubical structure $\Longrightarrow$ $d^{n+1} d^n = 0$.

$H^n(S, \sigma; M) = \mathrm{Ker}\, d^n / \mathrm{Im}\, d^{n-1}$ is the <u>nth cohomology group</u> of (S, σ) with coefficients in M.

7 Braided cohomology

Data: $\mathcal{C} = \mathbf{Set}$, YBE solution (S, σ), bimodule M over it.

Construction:

$C^n(S, \sigma; M) = \mathrm{Maps}(S^{\times n}, M)$,

$d^n = \sum_{i=1}^{n+1} (-1)^{i-1} (d_l^{n;i} - d_r^{n;i}) \colon C^n \to C^{n+1}$.

Theorems:

- This defines a precubical structure $\Longrightarrow$ $d^{n+1} d^n = 0$.
- For "nice" M (has a product compatible with the bimodule structure), there is a cup product on C^* inducing
$$\smile \colon H^n \otimes H^m \to H^{n+m},$$
which is graded commutative for "even nicer" M (trivial bimodule $\mathbb{k}$), with an explicit Steenrod-like homotopy.
- Other good properties.

My favourite proofs:

✓ graphical calculus;

✓ algebraic properties of quantum shuffles (*Rosso '95*).

8 A good theory?

1) & 2) For $\omega \in C^2(S,\sigma;\mathbb{Z})$,

$$d^2\omega = 0 \quad \Longleftrightarrow \quad \omega \text{ yields Boltzmann weights}$$

& diagonal deformations,

$$\omega = d^1\theta \quad \Longrightarrow \quad \omega \text{ yields trivial...}$$

3) Unifies classical cohomology theories.

Example: monoid $(S,*,1)$, $\sigma(x,y) = (1, x*y)$,

$$d_l^{n;i} f : \quad \ldots x_{i-2}\, \underline{x_{i-1}\, x_i}\, x_{i+1} \ldots \xrightarrow{\sigma_{i-1}}$$

$$\ldots \underline{x_{i-2}\, 1}\, (x_{i-1} * x_i)\, x_{i+1} \ldots \xrightarrow{\sigma_{i-2}}$$

$$\ldots \underline{1} x_{i-2}\, (x_{i-1} * x_i)\, x_{i+1} \ldots \longrightarrow \ldots$$

$$1\, x_1 \ldots x_{i-2}\, (x_{i-1} * x_i)\, x_{i+1} \ldots \longrightarrow$$

$$f(\ldots x_{i-2}\, (x_{i-1} * x_i)\, x_{i+1} \ldots).$$

4) Can be interpreted in terms of classifying spaces.

8 A good theory?

5) Quantum symmetriser $\mathcal{QS}$:

a subcomplex of braided complex for (S,σ) with coefs in M	$\xleftarrow{\mathcal{QS}}$	Hochschild complex for $\Bbbk\,\mathrm{Mon}(S,\sigma)$ with coefs in M
cup product		cup product
smaller complexes		tools

n=2: $\mathcal{QS}(f)(x,y) = f(x,y) - f(y',x')$, $\quad \sigma(x,y) = (y',x')$.

Open problem (*Yang, Farinati & García-Galofre '16*):
How far is $\mathcal{QS}$ from being a quasi-iso?

Theorem (*FGG '16*): $\mathcal{QS}$ is a q-iso when $\sigma^2 = \mathrm{Id}$, $\quad \mathrm{Char}\,\Bbbk = 0$,
the subcomplex is defined by

$$f(\ldots,x,y,\ldots) + f(\ldots,y',x',\ldots) = 0.$$

9 Cohomology: idempotent case

a subcomplex of braided complex for (S,σ) with coefs in M $\overset{\mathcal{QS}}{\longleftarrow}$ Hochschild complex for $\Bbbk\mathrm{Mon}(S,\sigma)$ with coefs in M

n=2: $\mathcal{QS}(f)(x,y) = f(x,y) - f(y',x')$, $\quad \sigma(x,y) = (y',x')$.

Open problem: How far is $\mathcal{QS}$ from being a quasi-iso?

Theorem (L '16): $\mathcal{QS}$ is a quasi-iso when $\sigma^2 = \sigma$ & the subcomplex is defined by

$$f(\ldots,x,y,\ldots) = 0 \qquad \text{whenever} \qquad \sigma(x,y) = (x,y),$$

i.e., f is supported on critical n-tuples only:

$$\mathrm{Crit}_n(S,\sigma) = \{(x_1,\ldots,x_n) \in S^{\times n} \mid \forall i,\ \sigma(x_i,x_{i+1}) \neq (x_i,x_{i+1})\}.$$

Proof: Algebraic discrete Morse theory.

9 Cohomology: idempotent case

Examples:

✓ Ordered set S, $\sigma(x,y)=(\min\{x,y\},\max\{x,y\})$,

$\mathsf{Mon}(S,\sigma)=\mathsf{Sym}(S)$,

$\mathsf{Crit}_n(S,\sigma)=\{x_1>x_2\ldots>x_n\}$

$\rightsquigarrow$ recovers the classical minimal resolution of $\Bbbk[S]$ & the resulting cohomology computations.

✓ Ordered set S, $\sigma(x,y)=(x,\max\{x,y\})$.

$\mathsf{Mon}(S,\sigma)=\langle\, S \,|\, \forall x>y,\ xy=xx\,\rangle$,

$\mathsf{Crit}_n(S,\sigma)=\{x_1>x_2\ldots>x_n\}$

$\rightsquigarrow$ improves *Jöllenbeck–Welker '09*.

9 Cohomology: idempotent case

Examples:

✓ Monoid factorisation $G = HK$,

$$S = H \cup K, \quad \sigma(x,y) = (h,k),\ h \in H,\ k \in K,\ hk = xy.$$

$$\mathrm{Mon}(S,\sigma)/_{(\star)} \simeq G, \qquad (\star): \quad 1_G = 1_{\mathrm{Mon}}$$

$$\mathrm{Crit}_n(S,\sigma) = \sqcup_{p+q=n} \overline{K}^{\times p} \times \overline{H}^{\times q}, \qquad \overline{K} = K \setminus \{1\},\ \overline{H} = H \setminus \{1\}$$

$\rightsquigarrow$ a double complex

$$C^{p,q} = \mathrm{Maps}(\overline{K}^{\times p} \times \overline{H}^{\times q}, M)$$

specialising to the Künneth formula
for the direct product $G = H \times K$.

9 Cohomology: idempotent case

Examples:

✓ Young tableaux on $\{1, \ldots, n\}$.

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2	6	6
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Robinson '38, Schensted' 61, Knuth '70: associative product $*$ on $\mathbf{YT}_n$.

Useful gadgets:

- ⇝ representation theory of S_n, $GL_n(\mathbb{C})$ and $GL_n(F_q)$;
- ⇝ intersections of Grassmannians;
- ⇝ products of symmetric functions;
- ⇝ lattice models;
- ⇝ crystal bases for quantum groups.

9 Cohomology: idempotent case

Examples:

✓ Young tableaux on $\{1, \dots, n\}$.

Theorem (*Cain et al., Bokut et al. '15*): One-column tableaux $\mathbf{Col}_n$ form a Gröbner–Shirshov basis for $(\mathbf{YT}_n, *)$.

Corollary (*Lopatkin '16*): First steps towards cohomology computations for $\Bbbk\mathbf{YT}_n$.

Work in progress: Idempotent YBE solutions σ_c on $\mathbf{Col}_n$ and σ_r on $\mathbf{Row}_n$ such that

$$(\mathbf{YT}_n, *) \simeq \mathrm{Mon}(\mathbf{Col}_n, \sigma_c)/_{(\star)} \simeq \mathrm{Mon}(\mathbf{Row}_n, \sigma_r)/_{(\star)}$$

Corollary: Manageable complexes computing the cohomology of $\Bbbk\mathbf{YT}_n$.