Structure groups & structure racks of YBE solutions

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① Introduction

X: set

$\sigma: X \times X \to X \times X$

$(a,b) \mapsto ({}^a b, a^b)$

$\boxed{\text{YBE}}$: $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ $\quad X^{\times 3} \to X^{\times 3}$

$\sigma_1 = \sigma \times Id_X$

$\sigma_2 = Id_X \times \sigma$

Solutions are interesting for
(1) statistical physics
(2) low-dimensional topology
(3) group theory
(4) rewriting
(5) homology theory
(6) non-commutative geometry
... and many more!

types of solutions
- $\exists \sigma^{-1}$, linear
- $\exists \sigma^{-1}$ & non-degenerate (ND) & $\sigma^2 \neq Id$
  - $a \cdot$ and $\cdot^b$ are bijections $X \xrightarrow{1:1} X$
- $\sigma^2 = Id$
- $\sigma^2 = \sigma$
- $\sigma^2 = Id$ or σ

Open pb: Classify solutions (of any type!).

Slogan: There are 3 axes in the world of YBE solutions:

$\boxed{\text{involutive}}$: $\sigma^2 = Id$

multi-permutation MP

permutation: $(f(b), f^{-1}(a))$ $\quad f: X \xrightarrow{1:1} X$

trivial: (b,a)

$\boxed{\text{idempotent}}$: $\sigma^2 = \sigma$ $\longleftrightarrow$ $(1, a\cdot b)$ $\overset{1:1}{\updownarrow}$ monoids $(X, \cdot, 1)$

$\boxed{\text{"half-trivial"}}$ $(b \lhd a, a)$ $\overset{1:1}{\longleftrightarrow}$ shelves: $(X, \lhd)$, s.t.

$(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$ (self-distributivity)

Ex: (1) X group, $a \lhd b = b^{-1} a b$

(2) $X \in \mathbb{Z}[t^{\pm 1}]$-Mod, $a \lhd b = ta + (1-t)b$

and many more!

rack: shelf s.t. all $\cdot \lhd b$ are bijections

Plan:

②&③ "Projections" $\quad$ YBE $\xrightarrow{\text{structure group}}$ Grp, $\quad$ YBE $\xrightarrow{\text{structure shelf}}$ Shelves

④ A result on structure groups of all YBE solutions obtained using structure racks:

YBE $\to$ Grp

YBE $\leftrightarrows$ Shelves

② YBE → Grp

YBE solution $(X, \sigma) \mapsto$ structure group $\boxed{SG(X,\sigma)} = \langle X \mid ab = {}^a b\, a^b \ \forall a,b \in X \rangle$

structure monoids & algebras are defined similarly.

Restrictions of this projection to our 3 axes are particularly nice.

(1) Grp $\hookrightarrow$ YBE $\xrightarrow{SG}$ Grp is almost the identity

$\circlearrowleft$ Id

$/ 1_X = 1_{SG(X,r)}$

(2) Thm (Manin, Gateva-Ivanova & Van den Bergh, Etingof-Schedler-Soloviev, Jespers-Okniński, Chouraqui, Bachiller-Cedó-Vendramin, ..., 80'...)

(X,σ) finite, involutive, non-degenerate $\Rightarrow$ its structure objects are nice:

- $SMon(X,\sigma)$ I-type, cacellative, Ore
- $SG(X,\sigma)$ solvable, Garside, Bieberbach $\Rightarrow$ torsionless
- $kSMon(X,\sigma)$ Koszul, noetherian, Cohen-Macaulay, Artin-Schelter regular
- $SG(X,\sigma)$ bi-orderable $\Leftrightarrow$ free abelian $\Leftrightarrow (X,\sigma)$ is trivial
  left-orderable $\Leftrightarrow$ poly-$\mathbb{Z}$ $\Leftrightarrow$ (X,σ) is MP.

(3) Thm (L.-Vendramin '19): $(X,\triangleleft)$ finite rack $\Rightarrow$ dichotomy for $SG(X,\sigma_\triangleleft) =: G$:

$\sigma_\triangleleft(a,b) = (b \triangleleft a, a)$

i) $\cong \mathbb{Z}^r$, $r = \# Orb(X,\triangleleft)$

ii) non-abelian, non-left-orderable, has torsion.

Questions: i) Characterise such racks?

- first properties: • $a \triangleleft b = a \triangleleft (b \triangleleft c)$
  • $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft b$

  $\Rightarrow$ useless for topology!

- complete description for $r=2$ (Bardakov-Nasybullov '19).

ii) Better understand such groups?

- some other properties: • $\mathbb{Z}$-graded
  • $Ab(G) \cong \mathbb{Z}^r$
  • $SG(G) \cong G \times \underbrace{K}_{\text{abelian}}$ (Ryder '93)

  seen as a YBE solution, with $(g,h) \mapsto (g^{-1}hg, g)$

  rmk: for any G, $SG(G) \cong G \times_\omega \underbrace{K}_{ab.}$, ω 2-cocycle

- virtually free abelian $\Rightarrow$ linear $\Rightarrow$ residually finite

$\mathbb{Z}^r \hookrightarrow G \twoheadrightarrow \overline{G}$ (finite)

$\underset{\text{basis}}{e_i} \mapsto a_i^{\,d}$ • a_i are orbit representatives of X
• d is well chosen.

③ YBE $\twoheadrightarrow$ Shelves (Etingof–Schedler–Soloviev '99, Lu–Yan–Zhu '00, Soloviev '00, L.–Vendramin '17).

LeftND YBE solution $(X,\sigma) \mapsto$ structure shelf $\boxed{SS(X,\sigma)} = (X, \triangleleft_\sigma)$, where $a \triangleleft b = {}^c a|^b$ & c satisfies $c^a = b$.

Prop.:
- $SS(X,\sigma)$ is a rack $\iff \exists \sigma^{-1}$
- $a \triangleleft_\sigma b = a \iff \sigma^2 = Id$
- $a \triangleleft_\sigma b = b \iff \sigma^2 = \sigma$.
- $a \triangleleft_\sigma a = a \iff$ biquandle.
- Shelves $\hookrightarrow$ YBE $\rightarrow$ Shelves, composite Id.

For this map the metaphor "projection" works particularly well!

Thm:
A birack and its structure rack yield the same braid invariants.
(birack = ND invertible YBE solution)

Cr!: A biquandle and its structure quandle yield the same knot invariant.
("birack & $\exists t: X \xrightarrow{1:1} X$ s.t. $\sigma(a, t(a)) = (a, t(a))$")

Thm': (X,σ) a biquandle $\Rightarrow$
$\boxed{\exists \text{ group 1-cocycle } J: SG(X,\sigma) \xrightarrow{1:1} SG(X, \triangleleft_\sigma)}$
$:= SG(YBE(SS(X,\sigma))$

Idem for structure monoids & algebras

Application (Cedó–Jespers–Okniński '19):
Lower & upper bounds for $\# SM_2(X,\sigma)$.
(the degree 2 component of the structure monoid)

④ Finite quotients of structure groups

<u>Thm</u>: (X,σ) finite biquandle, $G = SG(X,\sigma) \Rightarrow$

- $\boxed{\exists\ \mathbb{Z}^r \hookrightarrow G \twoheadrightarrow \underset{\text{finite}}{\overline{G}}}$ $\quad r = \#\,\mathrm{Orb}(SS(X,\sigma))$

- $$\boxed{\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "\bar\iota"'] & G \arrow[d] \\ & \overline{G} \end{tikzcd} \quad \begin{array}{c} \iota \text{ injective} \\ \Updownarrow \\ \bar\iota \text{ injective} \end{array}}$$
  $\iota(a) := a$

<u>Cor</u>: (1) a finitary setting for testing injectivity
$\Downarrow$
(X,σ) has nice properties

(2) G is virtually free abelian $\Rightarrow \ldots$

(3) $\boxed{G \text{ bi-orderable} \Leftrightarrow G \cong \mathbb{Z}^r.}$

<u>History</u>: $\left.\begin{array}{l} \text{• Chouraqui-Godelle '14} \\ \text{• Dehornoy '15} \end{array}\right] \sigma^2 = \mathrm{Id}$

- L.-Vendramin '19

<u>Idea of proof</u>: $J : SG(X,\sigma) \xrightarrow{1:1} SG(X, \triangleleft_\sigma)$

$a_i^{[d]} \longleftarrow a_i^d$

<u>Open pb</u>: $SG(X,\sigma)$ left-orderable $\Leftrightarrow$?