Structure groups of YBE solutions: cohomological applications

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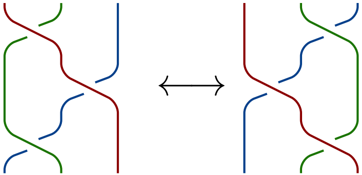
Spa, June 2017

$$(ab)c = a(bc)$$

3		
2	6	6
1	4	5



$$z^{-1}(y^{-1}xy)z = (z^{-1}y^{-1}z)(z^{-1}xz)(z^{-1}yz)$$

1 Yang–Baxter equation

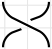
Data:

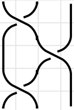
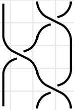
- monoidal category $\mathcal{C}$ (= $\mathbf{Vect}_{\mathbb{k}}$);
- object S;
- morphism $r \colon S \otimes S \to S \otimes S$.

braiding

YBE: $\boxed{r_1 r_2 r_1 = r_2 r_1 r_2 \colon S^{\otimes 3} \to S^{\otimes 3}}$ $r_1 = r \otimes \mathrm{Id}_S,\ r_2 = \mathrm{Id}_S \otimes r$

Topological avatar:

$r \longleftrightarrow$  

YBE $\longleftrightarrow$  $=$ 

Reidemeister III move

2 YBE zoology

We'll mostly work with set-theoretic solutions: $\mathcal{C} = \mathbf{Set}$ (*Drinfel'd* '90).

$\overset{\text{linearise}}{\rightsquigarrow} \quad \overset{\text{deform}}{\rightsquigarrow}$ linear solutions.

Example: $r(x, y) = (y, x)$

$\rightsquigarrow$ R-matrices;

$\rightsquigarrow$ $r_{\mathrm{Lie}}(x \otimes y) = y \otimes x + \hbar 1 \otimes [x, y]$:

$$\text{YBE for } r_{\mathrm{Lie}} \underset{1 \text{ central}}{\iff} \text{Jacobi for } [\,]$$

Example: $r_{SD}(x, y) = (y, x \triangleleft y)$:

$$\text{YBE for } r_{SD} \iff \text{self-distributivity for } \triangleleft$$

Self-distributivity: $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$

3 Self-distributivity

Example: $r_{SD}(x, y) = (y, x \lhd y)$:

YBE for r_{SD} $\iff$ self-distributivity for $\lhd$

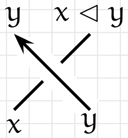
Self-distributivity: $(x \lhd y) \lhd z = (x \lhd z) \lhd (y \lhd z)$

Examples:

- group S with $x \lhd y = y^{-1}xy$;
- abelian group S, $t\colon S \to S$, $a \lhd b = ta + (1-t)b$.

Applications:

- invariants of knots and knotted surfaces (*Joyce & Matveev* '82);



- Hopf algebra classification (*Andruskiewitsch–Graña* '03).

4 Two axes

Example: Involutive solutions r, i.e., $r^2 = \mathrm{Id}_{S\times S}$.

A solution $r(a,b) = (\sigma_a(b), \tau_b(a))$ is called left non-degenerate (LND) if the maps τ_b are bijective.

Theorem (*Rump* '04): LND involutive solutions $\overset{1:1}{\longleftrightarrow}$ cycle sets.

Theorem (*Soloviev & Lu–Yan–Zhu* '00, *L.–Vendramin* '17):

- LND solution $(S, r) \rightsquigarrow$ SD operation $\triangleleft_r$ on S;
- $\triangleleft_r$ captures major properties of r; for instance,

$$r^2 = \mathrm{Id}_{S\times S} \iff a \triangleleft_r b = a.$$

So, involutive and self-distributive solutions can be seen as two perpendicular axes in the space of all LND solutions. Schematically,

$$\text{"}0 \to \mathbf{CycleSets} \to \mathbf{LNDSol} \to \mathbf{SD} \to 0\text{"}$$

5 Getting more exotic

We will tolerate non-invertible solutions.

Example: free self-distributive structures.

Application: total order on braid groups (*Dehornoy* '91).

Even worse: some of our solutions are idempotent: $rr = r$.

Examples: ✓ Monoid $(S, \cdot, 1)$, $r_{Ass}(x, y) = (1, x \cdot y)$:

$$\text{YBE for } r_{Ass} \underset{1 \text{ unit}}{\iff} \text{associativity for } \cdot$$

✓ Factorised monoid $G = HK$,
$S = H \cup K$, $r_{Fact}(x, y) = ((xy)_H, (xy)_K)$.

✓ Lattice $(S, \bigwedge, \bigvee)$, $r_L(x, y) = (x \bigwedge y, x \bigvee y)$.

6 Abstract nonsense?

So, YBE provides a unifying framework for many algebraic situations.

Question: Can anything non-trivial be done in such a general setting?

Answer: Yes!

1) A study of structure groups of solutions.
2) A (co)homology theory.

7 Why should a group theorist care about YBE?

Structure group, or universal enveloping group of (S, r):

$$G(S, r) = \langle\, S \mid xy = y'x' \text{ whenever } r(x, y) = (y', x') \,\rangle$$

Structure monoids and algebras are defined similarly.

$$G(S, r, e) := {}^{G(S,r)}/_{e = 1}$$

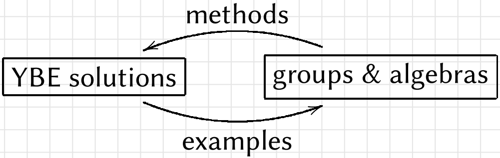
Examples:

✓ Factorised monoid $G = HK$, $r_{\mathrm{Fact}}(x, y) = ((xy)_H, (xy)_K)$:
$\mathrm{Mon}(H \cup K, r_{\mathrm{Fact}}, 1_G) \simeq G$.

✓ Lie algebra V, $V' = V \oplus \Bbbk 1$, 1 central, $r_{\mathrm{Lie}}(x \otimes y) = y \otimes x + 1 \otimes [x, y]$:
$\mathrm{Alg}(V', r_{\mathrm{Lie}}, 1) \simeq \mathrm{UEA}(V, [\,])$.

Why should a group theorist care about YBE?

$$G(S,r) = \langle\, S \mid xy = y'x' \text{ whenever } r(x,y) = (y',x') \,\rangle$$



Strategy (*Cedó–Jespers–del Río* '10):

Step 1: classify all structure groups G (or certain quotients thereof);

Step 2: classify all YBE solutions with $G(S,r) \cong G$.

Theorem: $r^2 = \mathrm{Id} \implies$

✓ $\mathrm{Mon}(S,r)$ is of I-type, cancellative, Ore;

✓ $\mathrm{Grp}(S,r)$ is solvable, Garside, Bieberbach;

✓ $\Bbbk\,\mathrm{Mon}(S,r)$ is Koszul, noetherian, Cohen–Macaulay, Artin–Schelter regular

(*Manin, Gateva-Ivanova & Van den Bergh, Etingof–Schedler–Soloviev, Jespers–Okniński, Chouraqui 80'-…*).

8 Braided cohomology

Construction (*Fenn et al.* '93, *Carter et al.* '04, *L.* '13):

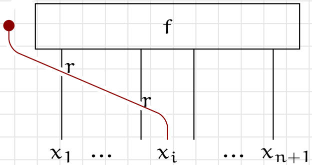
✓ $C^n := \mathrm{Maps}(S^{\times n}, \mathbb{Z}_m)$;
✓ $d^n \colon C^n \to C^{n+1}$, $\quad d^n = \sum_{i=1}^{n+1} (-1)^{i-1} (d_l^{n;i} - d_r^{n;i})$.

Versions:

✓ diagrammatic:

$$f(x'_1 \ldots x'_{i-1} x_{i+1} \ldots x_{n+1})$$
$$\uparrow$$
$$x'_i x'_1 \ldots x'_{i-1} x_{i+1} \ldots x_{n+1}$$
$$r_1 \cdots r_{i-1} \uparrow$$
$$x_1 \ldots x_{n+1}$$

$d_l^{n;i} f =$



✓ a topological realisation;

✓ using quantum shuffles;

✓ using a differential graded bialgebra (*Farinati–García-Galofre* '16).

9 Why I like braided cohomology

① Describes diagonal deformations (*Freyd–Yetter* '89, *Eisermann* '05):

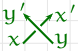
$$r_\omega(x,y) = q^{\omega(x,y)} r(x,y), \ \omega\colon S \times S \to \mathbb{Z}_m.$$

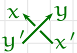
$d^2\omega = 0 \quad \Longrightarrow \quad r_\omega$ is a YBE solution.

② Yields knot and knotted surface invariants (*Carter et al.* '01):

(S, r)-coloured diagram $(D, \mathcal{C})$ & $\omega\colon S \times S \to \mathbb{Z}_m$

$\rightsquigarrow$ Boltzmann weight $\mathscr{B}_\omega(\mathcal{C}) = \sum \omega(x,y) - \sum \omega(x,y).$





$d^2\omega = 0 \quad \Longrightarrow \quad \sum_{\mathcal{C}} t^{\mathscr{B}_\omega(\mathcal{C})}$ is a knot invariant;

$\omega - \omega' = d^1\psi \quad \Longrightarrow \quad \omega$ and ω' yield equivalent invariants.

9 Why I like braided cohomology

③ Unifies cohomology theories for

- ✓ self-distributive structures $r_{SD}(x,y) = (y, x \lhd y)$
- ✓ associative structures $r_{Ass}(x,y) = (1, x \cdot y)$
- ✓ Lie algebras $r_{Lie}(x \otimes y) = y \otimes x + \hbar 1 \otimes [x,y]$

....

+ explains parallels between them,

+ suggests theories for new structures:

Example: cycle sets and braces (*L.–Vendramin* '17).

④ Computes the cohomology of structure groups.

10 Comparing cohomologies

Quantum symmetriser $\mathcal{QS}$:

braided cohomology		Hochschild cohomology
$H^*(S, \mathbb{Z}_m)$	$\overset{\mathcal{QS}}{\longleftarrow}$	$HH^*(\mathrm{Mon}(S,r), \mathbb{Z}_m)$
cup product $\smile$		cup product $\smile$
small complexes		tools

Theorem: $\mathcal{QS}$ is an isomorphism when

- ✓ $rr = \mathrm{Id}$ (*Farinati–García-Galofre* '16);
- ✓ $rr = r$ (*L.* '17).

Open question: For general r?

Applications:

- ✓ Spectral sequence for factorised monoids $G = HK$.
- ✓ Cohomology computations for plactic monoids.