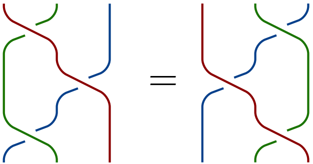
Unexpected facets of the Yang–Baxter equation

Victoria LEBED



$$(ab)c = a(bc)$$

$$z^{-1}(y^{-1}xy)z = (z^{-1}y^{-1}z)(z^{-1}xz)(z^{-1}yz)$$

TCD, Octobre 2016

1 Yang–Baxter equation: basics

Data: vector space V, $\sigma : V^{\otimes 2} \to V^{\otimes 2}$.

Yang-Baxter equation (YBE)

$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 : V^{\otimes 3} \to V^{\otimes 3}$ $\qquad$ $\sigma_1 = \sigma \otimes \mathrm{Id}_V,\ \sigma_2 = \mathrm{Id}_V \otimes \sigma$

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Avatars:

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➺ quantum inverse scattering method for completely integrable systems (*Faddeev et al. '79*);

➺ factorisable S-matrices in 2-dim. quantum field theory (*Zamolodchikov '79*);

➺ R-matrices in quantum groups (*Drinfel'd 80'*);

➺ C^* algebras (*Woronowicz 80'*);

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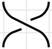
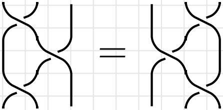
Avatars:

➜ braid equation in low-dimensional topology

$\sigma \longleftrightarrow$ 

YBE $\longleftrightarrow$ 

Reidemeister III move

2 Braided sets

Data: set S, $\sigma\colon S^{\times 2} \to S^{\times 2}$.

Set-theoretic YBE (*Drinfel′d '90*)

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- ✓ Lie algebra $(V, [\,])$, central element $1 \in V$,
  $\sigma(x \otimes y) = y \otimes x + \hbar 1 \otimes [x,y]$.

YBE for σ $\iff$ Jacobi identity for $[\,]$

3 Self-distributivity

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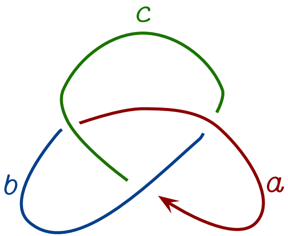
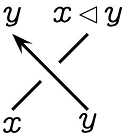
Applications:

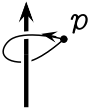
⇝ invariants of knots and knotted surfaces
(*Joyce & Matveev '82*);

$(S, \triangleleft)$-colourings of knot diagrams:





cf. Wirtinger presentation of $\pi_1(\mathbb{R}^3 \setminus K)$:



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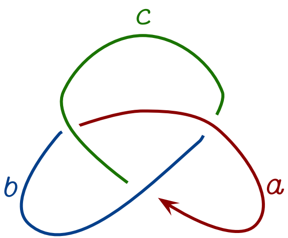
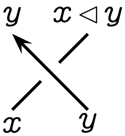
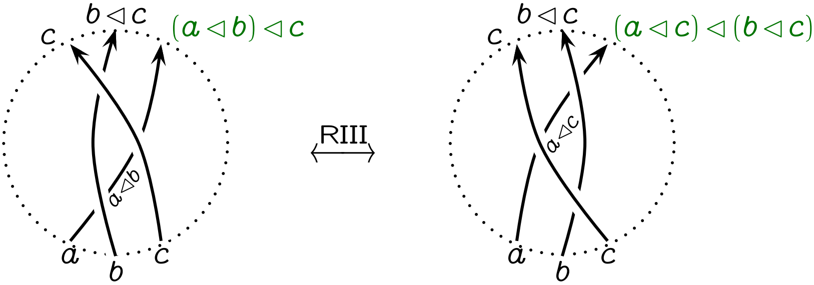




Proposition: $(S, \triangleleft)$ is a quandle $\Longrightarrow$ $\#\{\,(S, \triangleleft)\text{-colourings of diagrams}\,\}$ is a knot invariant.



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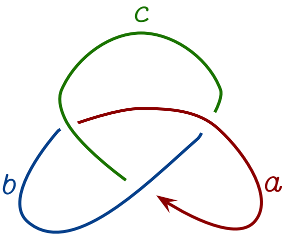

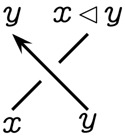

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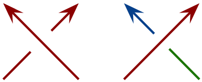

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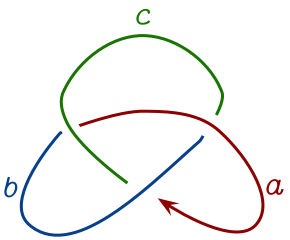

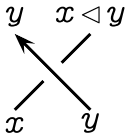

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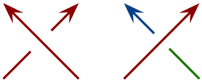

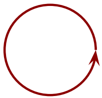

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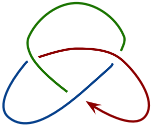

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- Hopf algebra classification (*Andruskiewitsch–Graña '03*).

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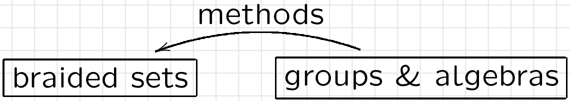
All these braidings are idempotent: $\sigma\sigma = \sigma$.

5 Universal enveloping constructions

Universal enveloping monoids:

$$\text{Mon}(S,\sigma)=\langle\, S \mid xy=y'x' \text{ whenever } \sigma(x,y)=(y',x')\,\rangle$$

U. e. (semi)groups and algebras are defined similarly.



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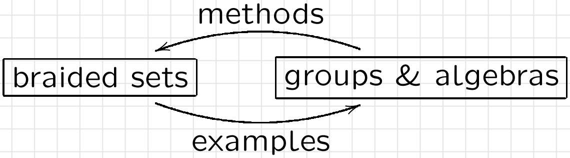

Theorem: (S,σ) a "nice" finite braided set, $\sigma^2 = \mathrm{Id} \Longrightarrow$

- ✓ $\mathrm{Mon}(S,\sigma)$ is of I-type, cancellative, Ore;
- ✓ $\mathrm{Grp}(S,\sigma)$ is solvable, Garside;
- ✓ $\Bbbk\mathrm{Mon}(S,\sigma)$ is Koszul, noetherian, Cohen–Macaulay, Artin–Schelter regular

(*Manin, Gateva-Ivanova & Van den Bergh, Etingof–Schedler–Soloviev, Jespers–Okniński, Chouraqui 80'-...*).

Example: $S = \{a, b\}$, $aa \overset{\sigma}{\longleftrightarrow} bb$, $ab \circlearrowleft \sigma$, $ba \circlearrowleft \sigma$;

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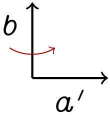
Realisation by Euclidean transformations of $\mathbb{R}^2$:



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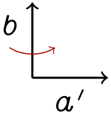

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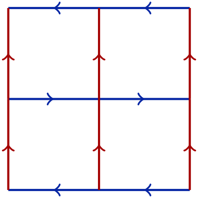
$\mathbb{R}^2/G \cong$ Klein bottle:



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✓ Lie algebra $(V, [], 1)$, $\sigma(x \otimes y) = y \otimes x + \hbar 1 \otimes [x,y]$,

$$UEA(V, []) \simeq {}^{\Bbbk \mathrm{Mon}(S,\sigma)}/_{1 = 1_{\mathrm{Mon}}}.$$

6 Representations

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Representations of (S,σ) := representations of $\Bbbk\,\mathrm{Mon}(S,\sigma)$,

i.e. vector spaces M with $M \times S \xrightarrow{\cdot} M$ s.t.

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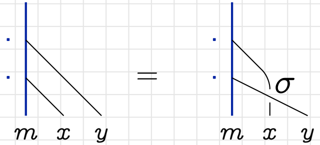

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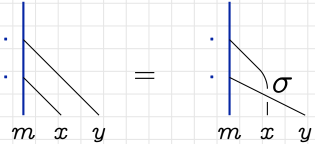

Examples:

- trivial rep.: $M = \Bbbk$, $m \cdot x = m$;
- $M = \Bbbk\,\mathrm{Mon}(S,\sigma)$, $m \cdot x = mx$;
- usual reps for monoids, Lie algebras, self-distributive structures.

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A cohomology theory for YBE solutions should:

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First approximation: diagonal deformations

$$\sigma_q(x,y) = q^{\omega(x,y)}\sigma(x,y),\ \omega\colon S \times S \to \mathbb{Z}.$$

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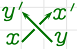
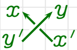
2) Yield knot and knotted surface invariants:

(S,σ)-coloured diagram $(D,\mathcal{C})$ & $\omega\colon S\times S \to \mathbb{Z}$

$\rightsquigarrow$ Boltzmann weight $\mathcal{B}_\omega(\mathcal{C}) = \sum \omega(x,y) - \sum \omega(x,y)$.





ω a 2-cocycle $\Longrightarrow$ a knot invariant given by

$$\{\mathcal{B}_\omega(\mathcal{C})\,|\,\mathcal{C} \text{ is a } (S,\sigma)\text{-colouring of } D\}.$$

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4) Compute the cohomology of $\Bbbk\mathrm{Mon}(S,\sigma)$.

8 Braided cohomology

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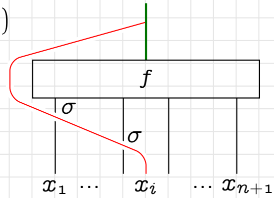

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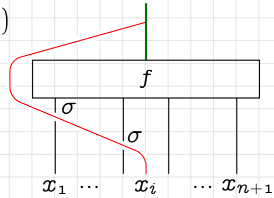

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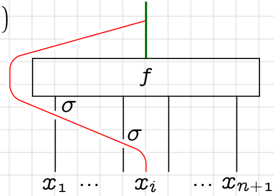

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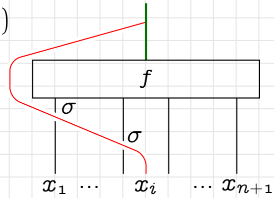

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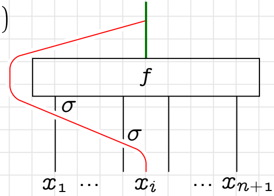

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Open problem: How far is $\mathcal{QS}$ from being an iso in general?