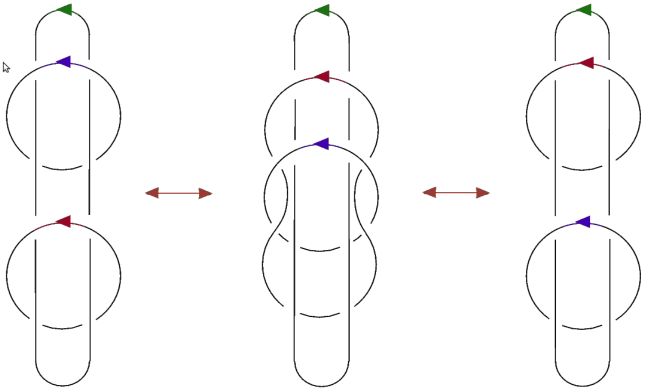
On set-theoretic solutions to the Yang-Baxter equation

Victoria LEBED (Nantes)
with Leandro VENDRAMIN (Buenos Aires)



Turin, January 2016

1 Yang-Baxter equation

✓ X: vector space,
✓ $\sigma\colon X^{\otimes 2} \to X^{\otimes 2}$.

Yang-Baxter equation (YBE)

$$\sigma_1 \circ \sigma_2 \circ \sigma_1 = \sigma_2 \circ \sigma_1 \circ \sigma_2 \colon X^{\otimes 3} \to X^{\otimes 3}$$

where $\sigma_1 = \sigma \otimes \mathrm{Id}_X$, $\sigma_2 = \mathrm{Id}_X \otimes \sigma$.

Origins:

➻ factorization condition for the dispersion matrix in the 1-dim. n-body problem (*McGuire & Yang 60'*);

➻ partition function for exactly solvable lattice models (*Baxter 70'*).

1 Set-theoretic Yang-Baxter equation

✓ X: set,
✓ $\sigma : X^{\times 2} \to X^{\times 2}$

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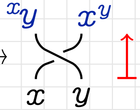
$\sigma \longleftrightarrow$



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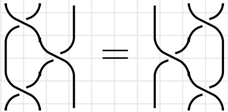

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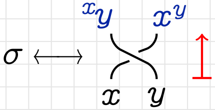

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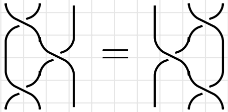

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Left non-degenerate braided set:
$x \mapsto x^{y}$ is a bijection $X \to X$
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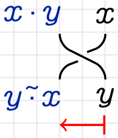

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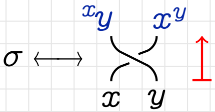

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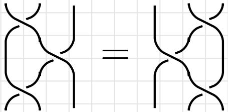

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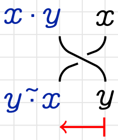

Birack: left and right non-degenerate braided set with invertible σ.

2 Structure (semi)group

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Structure (semi)group of (X, σ): $(S)G_{X,\sigma} = \langle X \mid xy = {}^{x}y\, x^{y} \rangle$

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⇝ Captures properties of σ.

⇝ A source of interesting groups and algebras:
Theorem: If (X,σ) is a finite RI-compatible birack with $\sigma^2 = \mathrm{Id}$, then

✓ $SG_{X,\sigma}$ is of I-type, cancellative, Öre;
✓ $G_{X,\sigma}$ is solvable, Garside;
✓ $\Bbbk SG_{X,\sigma}$ is Koszul, noetherian, Cohen-Macaulay, Artin-Schelter regular

(*Manin, Gateva-Ivanova & Van den Bergh, Etingof-Schedler-Soloviev, Jespers-Okniński, Chouraqui 80'-. . .*).

3 Self-distributive structures

Shelf: set X & $\lhd\colon X \times X \to X$ s.t.

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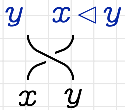
$\Leftrightarrow \sigma_{\lhd} =$ 

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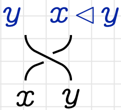
$\Leftrightarrow \sigma_{\triangleleft} =$ 

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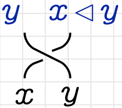
$\Leftrightarrow \sigma_{\triangleleft} =$ 

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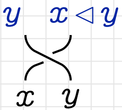
$\Leftrightarrow \sigma_{\lhd}=$ 

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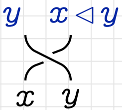
$\Leftrightarrow\ \sigma_\lhd=$ 

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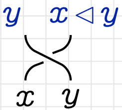
$\Leftrightarrow \sigma_{\triangleleft} =$ 

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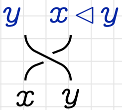

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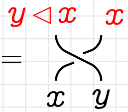
$\Leftrightarrow$ $\sigma_{\triangleleft}{}' =$ 

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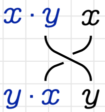

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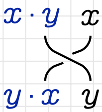
$\Leftrightarrow \sigma_{\cdot} =$ 

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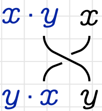

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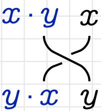

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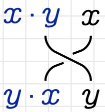
$\Leftrightarrow \sigma_{\cdot} =$ 

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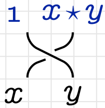
$\Leftrightarrow \sigma_\star =$ 

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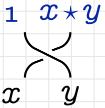
the associativity of $\star$ $\Leftrightarrow$ $\sigma_\star =$ 

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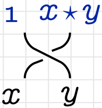
$\Leftrightarrow$ $\sigma_\star =$ 

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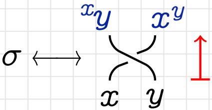
One has a semigroup isomorphism

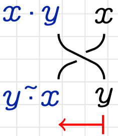
$$SG_{X,\sigma_\star} \xrightarrow{\sim} X,$$

$$X^{\times k} \ni x_1 \cdots x_k \mapsto x_1 \star \cdots \star x_k.$$

6 Associated shelf

Fix a LND braided set (X, σ).





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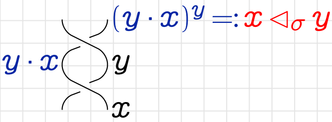

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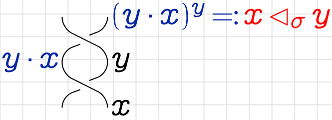
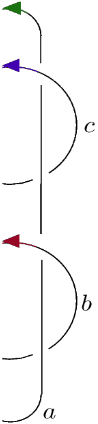


Proof:



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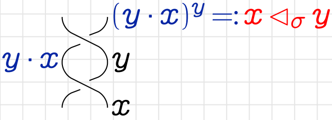

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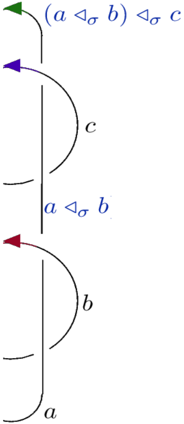
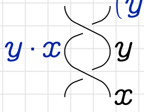


6 Associated shelf

Fix a LND braided set (X, σ).

Proposition (*L.-V. 2015*): one has a shelf $(X, \lhd_\sigma)$, where

$$(y \cdot x)^y =: x \lhd_\sigma y$$



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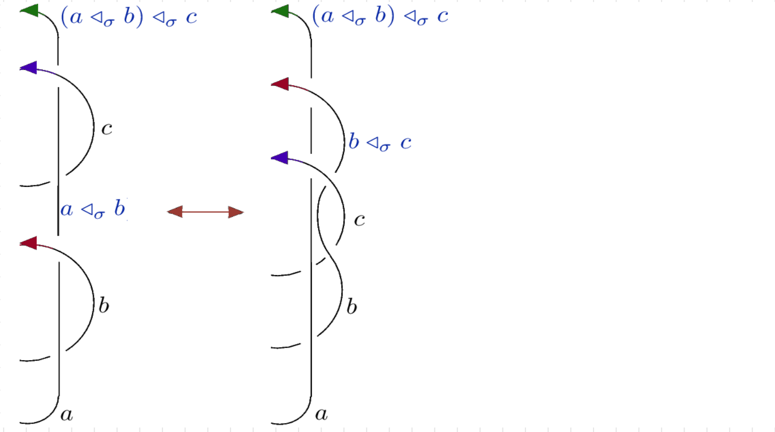

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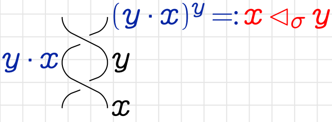

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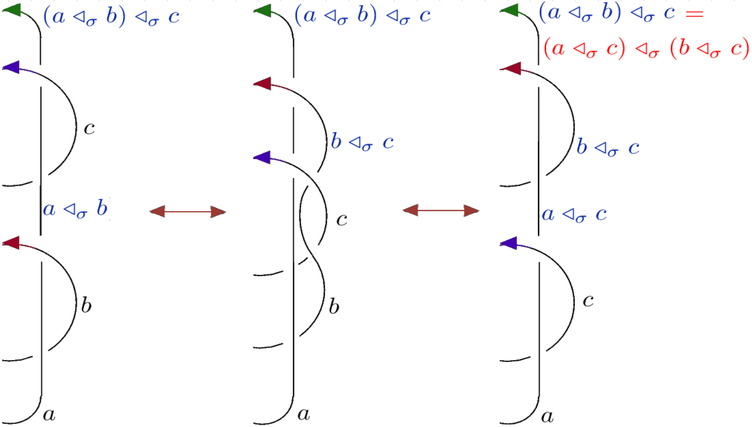

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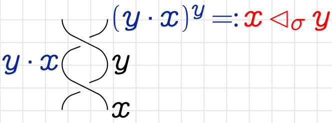

Motto: $\triangleleft_\sigma$ is often simpler than σ,
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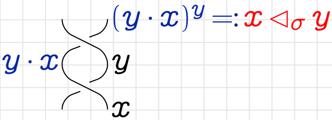


<u>Motto</u>: $\lhd_\sigma$ is often simpler than σ,
and encodes its important properties.

Proposition (*L.-V. 2015*):

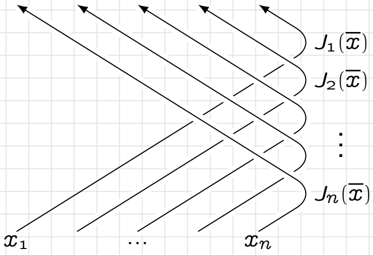
- $(X, \lhd_\sigma)$ is a rack $\Leftrightarrow$ σ is invertible;
- $(X, \lhd_\sigma)$ is a trivial ($x \lhd_\sigma y = x$) $\Leftrightarrow$ $\sigma^2 = \mathrm{Id}$;
- $x \lhd_\sigma x = x$ $\Leftrightarrow$ $\sigma(x \cdot x, x) = (x \cdot x, x)$.

7 Guitar map

$$J^{(n)} \colon X^{\times n} \xrightarrow{\sim} X^{\times n},$$

$$(x_1, \dots, x_n) \mapsto (x_1^{x_2 \cdots x_n}, \dots, x_{n-1}^{x_n}, x_n),$$

$$\text{where } x_i^{x_{i+1} \cdots x_n} = (\dots(x_i^{x_{i+1}})\dots)^{x_n}.$$



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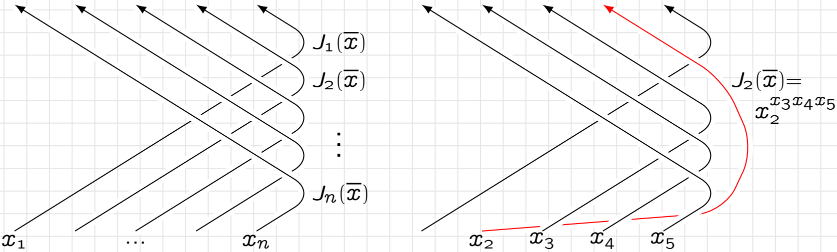

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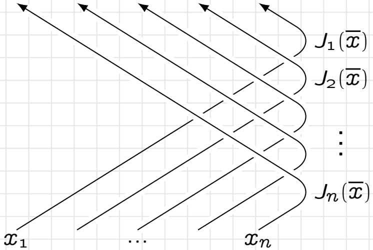

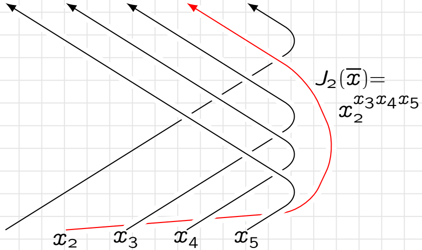

Proposition (*L.-V. 2015*): $J\sigma_i = \sigma_i' J$, where $\sigma' = \sigma'_{\triangleleft_\sigma}$.

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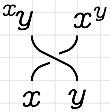

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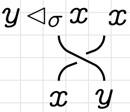

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$$J^{(n)}\colon X^{\times n} \xrightarrow{\sim} X^{\times n},$$

$$(x_1, \ldots, x_n) \mapsto (x_1^{x_2 \cdots x_n}, \ldots, x_{n-1}^{x_n}, x_n),$$

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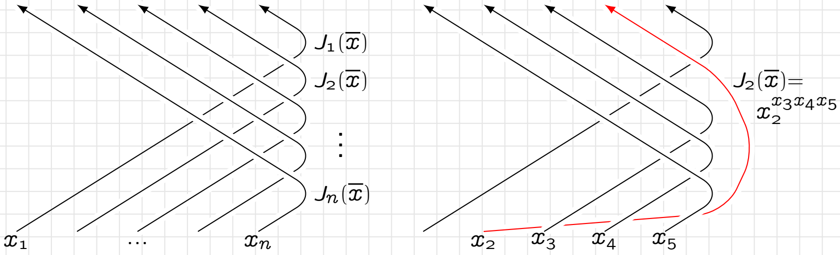

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<u>Corollary</u>: σ and σ' yield isomorphic B_n-actions on $X^{\times n}$.

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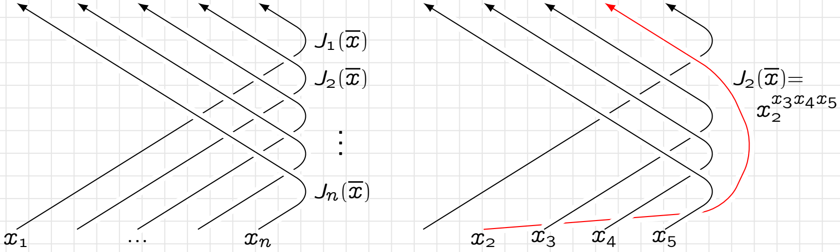

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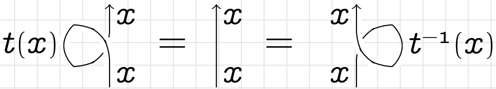
Warning: In general, $(X, \sigma) \not\cong (X, \sigma')$ as braided sets!

8 RI-compatibility

RI-compatible braiding: $\exists t\colon X \xrightarrow{\sim} X$ s.t. $\sigma(t(x),x) = (t(x),x)$.



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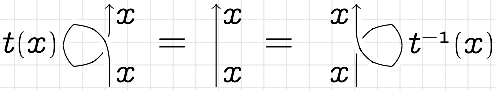

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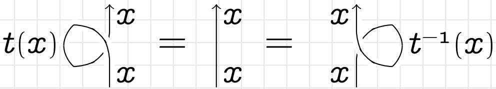

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Examples:

- for a rack, it means $x \triangleleft x = x$ (here $t(x) = x$);
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9 Structure group via associated shelf

Theorem (*L.-V. 2015*): (1) The guitar maps induce a bijective 1-cocycle $\boxed{J\colon SG_{X,\sigma} \xrightarrow{\sim} SG_{X,\sigma'}}$, where $\sigma' = \sigma'_{\lhd_\sigma}$.

Reminder: $SG_{X,\sigma} = \langle X \mid xy = {}^{x}y x^{y} \rangle$.

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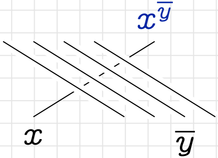

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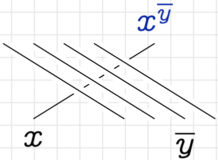

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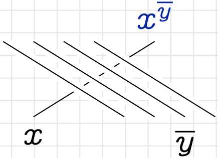

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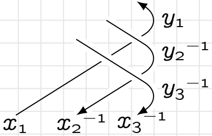

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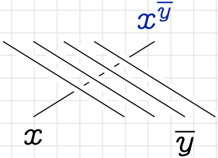

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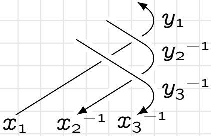

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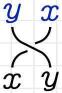
- ➛ $J\colon \sigma_\cdot \leftrightarrow$ flip 

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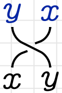
- $J\colon\ \sigma_\cdot \leftrightarrow$ flip 

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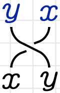
- $J\colon \sigma_\cdot \leftrightarrow$ flip 

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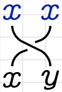
- $J\colon \sigma_\star \leftrightarrow$ 

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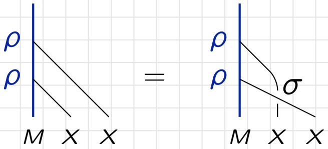
- $SG_{X,\sigma'_\star} \xrightarrow{\sim} X, \qquad x_1 \cdots x_k \mapsto x_1$.

11 Braided modules

Right braided module over (X,σ):

set M & ρ: $M \times X \to M$ s.t.



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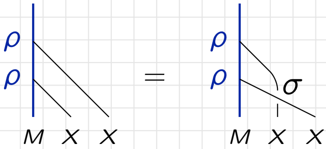

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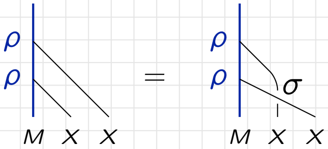

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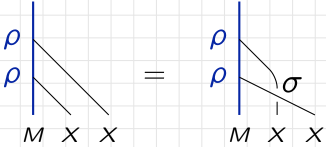

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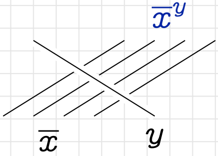
- adjoint modules: $M = X^{\times k}$, $\rho(\overline{x}, y) = \overline{x}^y$.



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Ingredients:

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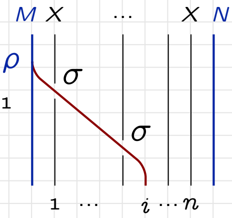
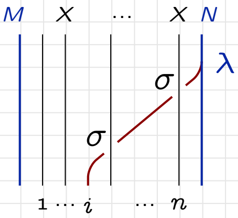
Theorem (*Carter-Elhamdadi-Saito 2004, L. 2013*):
C_k carry a family of differentials $\delta^{(\alpha,\beta)} = \alpha\, {}^{\bullet}\delta + \beta\, \delta^{\bullet}$, $\alpha, \beta \in \mathbb{Z}$.

${}^{\bullet}\delta = \sum (-1)^{i-1}$ 

$\delta^{\bullet} = \sum (-1)^{i-1}$ 

Braided homology.v2

Theorem (*Fenn-Rourke-Sanderson 1992, L.-V. 2015*):
If moreover σ is LND, then C_k carry a second family of differentials $\widehat{\delta}^{(\alpha,\beta)} = \alpha \,\widehat{{}^{\bullet}\delta} + \beta\, \widehat{\delta^{\bullet}}$, $\alpha, \beta \in \mathbb{Z}$, where

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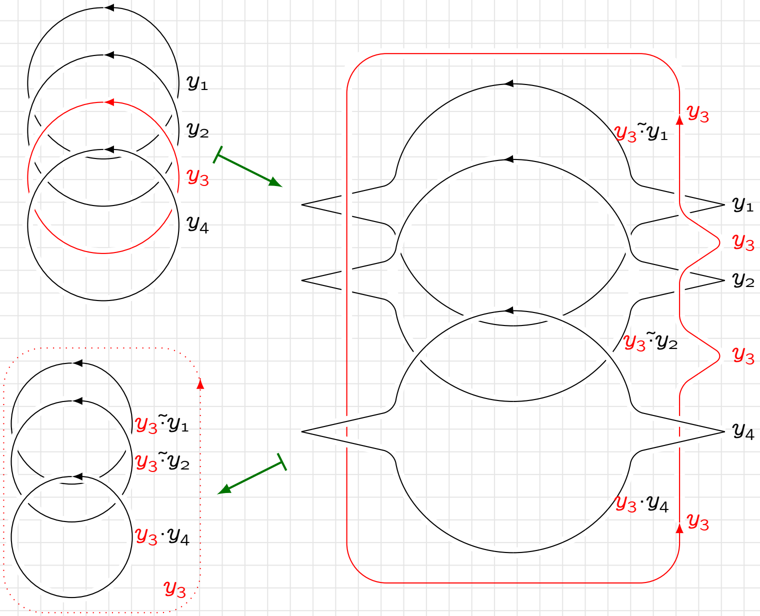
$$\widehat{\delta^{\bullet}}(m, \overline{y}, n) = \sum (-1)^{i-1} (m, y_i \,\tilde{\cdot}\, y_1, \ldots, y_i \,\tilde{\cdot}\, y_{i-1}, y_i \cdot y_{i+1}, \ldots, y_i \cdot y_k, \lambda(y_i, n)).$$

$$(m, y_1, y_2, y_3, y_4, n) \mapsto (m, y_3 \tilde{\cdot} y_1, y_3 \tilde{\cdot} y_2, y_3 \cdot y_4, \lambda(y_3, n)).$$



Braided homology: v1 = v2

Theorem (*L.-V. 2015*): If σ is LND, then

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Examples:

✓ group $(X, \star, 1)$

$$\bullet\ \delta_i(m, \overline{x}, n) = \begin{cases} (\ldots, x_{i-1} \star x_i, \ldots), & i > 1, \\ (m \star x_1, \ldots) & i = 1; \end{cases}$$

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$\rightsquigarrow$ 2 forms of the bar complex

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- ${}^\bullet\delta$ $\rightsquigarrow$ 1-term self-distributive homology

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- $\widehat{^\bullet\delta} - \widehat{\delta^\bullet}$, $M = \{*\}$, $N = X$:

$H^1 \simeq$ central extensions