Yang–Baxter Equation, Knots, Cohomology: a Golden Triangle

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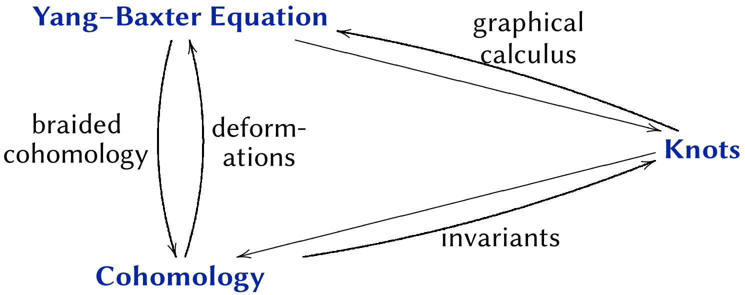

UCD, October 2017

1 Yang–Baxter equation: basics

Data: vector space V, $\sigma: V^{\otimes 2} \to V^{\otimes 2}$.

Yang–Baxter equation (YBE)

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 : V^{\otimes 3} \to V^{\otimes 3} \qquad \sigma_1 = \sigma \otimes \mathrm{Id}_V,\ \sigma_2 = \mathrm{Id}_V \otimes \sigma$$

Avatars:

- factorization condition for the dispersion matrix in the 1-dim. n-body problem (*McGuire & Yang 60'*);
- condition for the partition function in an exactly solvable lattice model (*Onsager '44; Baxter 70'*);
- quantum inverse scattering method for completely integrable systems (*Faddeev et al. '79*);
- factorisable S-matrices in 2-dim. quantum field theory (*Zamolodchikov '79*);
- R-matrices in quantum groups (*Drinfel′d 80'*);
- C^* algebras (*Woronowicz 80'*);

....

1 Yang–Baxter equation: basics

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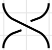
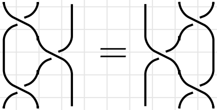
$\sigma_1 = \sigma \otimes \mathrm{Id}_V$, $\sigma_2 = \mathrm{Id}_V \otimes \sigma$

Avatars:

➻ braid equation in low-dimensional topology

$\sigma \longleftrightarrow$ 

YBE $\longleftrightarrow$ 

Reidemeister III move

2 Braided sets

Data: set S, $\sigma\colon S^{\times 2} \to S^{\times 2}$.

Set-theoretic YBE (*Drinfel′d ’90*)

$$\boxed{\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\colon S^{\times 3} \to S^{\times 3}} \qquad \sigma_1 = \sigma \times \mathrm{Id}_S,\ \sigma_2 = \mathrm{Id}_S \times \sigma$$

Solutions are called braided sets.

braided sets $\overset{\text{linearise}}{\rightsquigarrow}$ $\overset{\text{deform}}{\rightsquigarrow}$ general solutions

Examples:

✓ $\sigma(x, y) = (x, y)$;

✓ $\sigma(x, y) = (y, x)$ $\rightsquigarrow$ R-matrices;

✓ Lie algebra $(V, [\,])$, central element $1 \in V$,
$\sigma(x \otimes y) = y \otimes x + \hbar 1 \otimes [x, y]$.

$$\boxed{\text{YBE for } \sigma \iff \text{Jacobi identity for } [\,]}$$

3 Self-distributivity

✓ set S, binary operation $\lhd$, $\sigma(x,y) = (y, x \lhd y)$

YBE for σ $\iff$ self-distributivity for $\lhd$

Self-distributivity: $(x \lhd y) \lhd z = (x \lhd z) \lhd (y \lhd z)$

Examples:

➵ group S with $x \lhd y = y^{-1}xy$ yield a quandle: (SD)
& $\forall y$, $x \mapsto x \lhd y$ is a bijection
& $x \lhd x = x$;

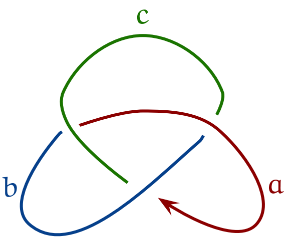
$$z^{-1}(y^{-1}xy)z = (z^{-1}y^{-1}z)(z^{-1}xz)(z^{-1}yz)$$

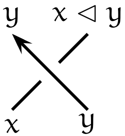
➵ abelian group S, $t\colon S \to S$, $a \lhd b = ta + (1-t)b$.

Applications:

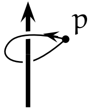
➵ invariants of knots and knotted surfaces (*Joyce & Matveev '82*);

$(S, \triangleleft)$-colourings of knot diagrams:



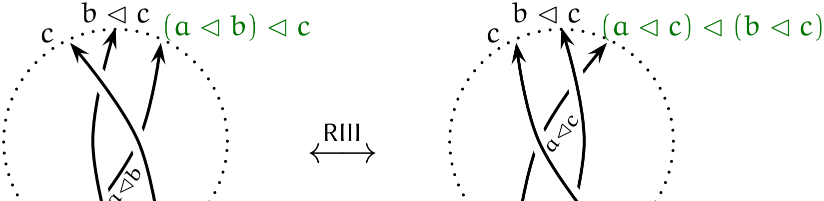


cf. Wirtinger presentation of $\pi_1(\mathbb{R}^3 \setminus K)$:



Proposition: $(S, \triangleleft)$ is a quandle $\implies$

$\#\{\, (S, \triangleleft)\text{-colourings of diagrams} \,\}$ is a knot invariant.



3 Self-distributivity

✓ set S, binary operation $\triangleleft$, $\sigma(x,y) = (y, x \triangleleft y)$

YBE for σ $\Longleftrightarrow$ self-distributivity for $\triangleleft$

Self-distributivity: $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$

Applications:

- invariants of knots and knotted surfaces (*Joyce & Matveev '82*);
- a total order on braid groups (*Dehornoy '91*);
- Hopf algebra classification (*Andruskiewitsch–Graña '03*);
- integration of generalised Lie algebras (*Kinyon '07*);
- study of braided sets (*L.–Vendramin '16*).

4 More examples of braided sets

✓ monoid $(S, *, 1)$, $\sigma(x, y) = (1, x * y)$;

$$\text{YBE for } \sigma \iff \text{associativity for } *$$

✓ monoid factorization $G = HK$,
$S = H \cup K$, $\sigma(x, y) = (h, k)$, $h \in H, k \in K, hk = xy$;

✓ lattice $(S, \bigwedge, \bigvee)$, $\sigma(x, y) = (x \bigwedge y, x \bigvee y)$.

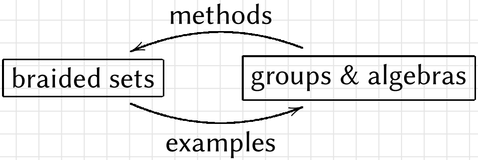
All these braidings are idempotent: $\sigma\sigma = \sigma$.

5 Universal enveloping constructions

Universal enveloping monoids:

$$\mathrm{Mon}(S,\sigma) = \langle\, S \mid xy = y'x' \text{ whenever } \sigma(x,y) = (y',x') \,\rangle$$

U. e. (semi)groups and algebras are defined similarly.



Theorem: (S,σ) a "nice" finite braided set, $\sigma^2 = \mathrm{Id} \implies$

- ✓ $\mathrm{Mon}(S,\sigma)$ is of I-type, cancellative, Ore;
- ✓ $\mathrm{Grp}(S,\sigma)$ is solvable, Garside;
- ✓ $\Bbbk\,\mathrm{Mon}(S,\sigma)$ is Koszul, noetherian, Cohen–Macaulay, Artin–Schelter regular

(*Manin, Gateva-Ivanova & Van den Bergh, Etingof– Schedler–Soloviev, Jespers–Okniński, Chouraqui 80'-…*).

5 Universal enveloping constructions

Universal enveloping monoids:

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Examples:

✓ monoid $(S,*,1)$, $\sigma(x,y) = (1, x*y)$,

$$S \simeq {}^{\mathrm{Mon}(S,\sigma)}/_{1_S = 1_{\mathrm{Mon}}};$$

✓ Lie algebra $(V,[\,],1)$, $\sigma(x\otimes y) = y\otimes x + \hbar 1\otimes [x,y]$,

$$\mathrm{UEA}(V,[\,]) \simeq {}^{\Bbbk\,\mathrm{Mon}(S,\sigma)}/_{1 = 1_{\mathrm{Mon}}}.$$

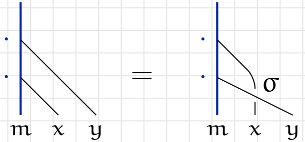
6 Representations

$$\mathrm{Mon}(S,\sigma) = \langle\, S \mid xy = y'x' \text{ whenever } \sigma(x,y) = (y',x') \,\rangle$$

Representations of (S,σ) := representations of $\Bbbk\,\mathrm{Mon}(S,\sigma)$,

i.e. vector spaces M with $M \times S \xrightarrow{\cdot} M$ s.t.

$$(m \cdot x) \cdot y \quad = \quad (m \cdot y') \cdot x'$$



Examples:

- trivial rep.: $M = \Bbbk$, $m \cdot x = m$;
- $M = \Bbbk\,\mathrm{Mon}(S,\sigma)$, $m \cdot x = mx$;
- usual reps for monoids, Lie algebras, self-distributive structures.

7 A cohomology theory?

A cohomology theory for YBE solutions should:

1) Describe deformations: $\sigma_0 \rightsquigarrow \sigma_0 + \hbar\sigma_1 + \hbar^2\sigma_2 + \cdots$.

Difficult! Pioneers: *Freyd–Yetter '89, Eisermann '05.*

First approximation: diagonal deformations

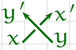
$$\sigma_q(x,y) = q^{\omega(x,y)}\sigma(x,y),\ \omega\colon S \times S \to \mathbb{Z}.$$

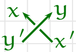
ω a 2-cocycle $\Longrightarrow$ σ_q a YBE solution.

2) Yield knot and knotted surface invariants (*Carter et al. '01*):

(S,σ)-coloured diagram $(D,\mathcal{C})$ & $\omega\colon S \times S \to \mathbb{Z}$

$\rightsquigarrow$ Boltzmann weight $\mathscr{B}_\omega(\mathcal{C}) = \sum \omega(x,y) - \sum \omega(x,y).$





ω a 2-cocycle $\Longrightarrow$ a knot invariant given by

$$\{\, \mathscr{B}_\omega(\mathcal{C}) \mid \mathcal{C} \text{ is a } (S,\sigma)\text{-colouring of } D \,\}.$$

7 A cohomology theory?

A cohomology theory for YBE solutions should:

3) Unify cohomology theories for
   - ➙ associative structures,
   - ➙ Lie algebras,
   - ➙ self-distributive structures etc.

+ explain parallels between them (*L. '13*),

+ suggest theories for new structures (*L.–Vendramin '16*).

4) Compute the cohomology of $\Bbbk\,\mathrm{Mon}(S, \sigma)$.

8 Braided cohomology

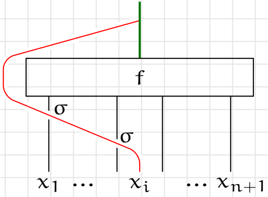
Data: braided set (S,σ) & bimodule M over it.

Construction:

$C^n(S,\sigma;M) = \mathrm{Maps}(S^{\times n}, M)$,

$d^n = \sum_{i=1}^{n+1} (-1)^{i-1}(d_l^{n;i} - d_r^{n;i})\colon C^n \to C^{n+1}$,

$$d_l^{n;i} f: \quad x_1 \ldots x_{n+1} \xrightarrow{\sigma_1 \cdots \sigma_{i-1}} x_i' x_1' \ldots x_{i-1}' x_{i+1} \ldots x_{n+1} \longrightarrow x_i' \cdot f(x_1' \ldots x_{i-1}' x_{i+1} \ldots x_{n+1})$$



Theorem: ➻ $d^{n+1} d^n = 0$;

$H^n(S,\sigma;M) = \mathrm{Ker}\, d^n / \mathrm{Im}\, d^{n-1}$ is the nth cohomology group of (S,σ) with coefficients in M;

➻ for "nice" M, a cup product $\smile\colon H^n \otimes H^m \to H^{n+m}$;

➻ other good properties.

9 A good theory?

1) & 2) For $\omega \in C^2(S, \sigma; \mathbb{Z})$,

$d^2\omega = 0 \implies \omega$ yields Boltzmann weights

& diagonal deformations,

$\omega = d^1\theta \implies \omega$ yields trivial...

3) Unifies classical cohomology theories.

Example: monoid $(S, *, 1)$, $\sigma(x, y) = (1, x * y)$,

$$d_l^{n;i} f: \quad \ldots x_{i-2}\, \underline{x_{i-1}\, x_i}\, x_{i+1} \ldots \xrightarrow{\sigma_{i-1}}$$

$$\ldots \underline{x_{i-2}\, 1}\, (x_{i-1} * x_i)\, x_{i+1} \ldots \xrightarrow{\sigma_{i-2}}$$

$$\ldots \underline{1} x_{i-2}\, (x_{i-1} * x_i)\, x_{i+1} \ldots \longrightarrow \ldots$$

$$1\, x_1 \ldots x_{i-2}\, (x_{i-1} * x_i)\, x_{i+1} \ldots \longrightarrow$$

$$f(\ldots x_{i-2}\, (x_{i-1} * x_i)\, x_{i+1} \ldots).$$

9 A good theory?

4) Quantum symmetriser $\mathcal{QS}$:

braided cohomology of (S, σ) with coefs in M	$\xleftarrow{\mathcal{QS}}$	Hochschild cohomology of $\Bbbk\,\mathrm{Mon}(S, \sigma)$ with coefs in M
cup product		cup product
smaller complexes		tools

$\mathcal{QS}$ is an isomorphism when

- $\sigma\sigma = \mathrm{Id}$ and $\mathrm{Char}\,\Bbbk = 0$ (*Farinati & García-Galofre '16*);
- $\sigma\sigma = \sigma$ (*L. '16*).

Applications: factorizable groups, Young tableaux.

Open problem: How far is $\mathcal{QS}$ from being an iso in general?