Associative Algebras, Bialgebras and Leibniz Algebras as Braided Objects

Victoria LEBED

Paris 7, IMJ

British Mathematical Colloquium
March 27, 2013

Plan

Braided categories

All categories are considered *strict monoidal* in this talk.

Definition

A category $\mathscr{C}$ is called *braided* if it is endowed with a *braiding*, i.e. a family of morphisms $c = \{c_{V,W} : V \otimes W \to W \otimes V\} \quad \forall V, W \in \mathrm{Ob}(\mathscr{C})$ which is

✓*natural*, i.e. for any $V, W, V', W' \in \mathrm{Ob}(\mathscr{C})$, $f \in \mathrm{Hom}_{\mathscr{C}}(V, V')$, $g \in \mathrm{Hom}_{\mathscr{C}}(W, W')$ one has

$$c_{V',W'} \circ (f \otimes g) = (g \otimes f) \circ c_{V,W}$$

✓*compatible with the tensor product*, i.e. $\forall V, W, U \in \mathrm{Ob}(\mathscr{C})$, one has

$$c_{V,W\otimes U} = (\mathrm{Id}_W \otimes c_{V,U}) \circ (c_{V,W} \otimes \mathrm{Id}_U),$$
$$c_{V\otimes W,U} = (c_{V,U} \otimes \mathrm{Id}_W) \circ (\mathrm{Id}_V \otimes c_{W,U}).$$

Braided categories

Definition

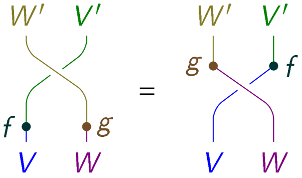
A category $\mathscr{C}$ is called *braided* if it is endowed with a *braiding*
$c = \{c_{V,W} : V \otimes W \to W \otimes V\} \quad \forall V, W \in \mathrm{Ob}(\mathscr{C})$ which is

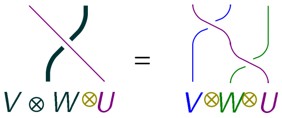
✓*natural*: $\quad c_{V',W'} \circ (f \otimes g) = (g \otimes f) \circ c_{V,W}$

✓*compatible with* $\otimes$:

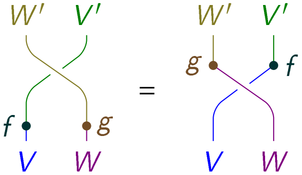
$$c_{V,W\otimes U} = (\mathrm{Id}_W \otimes c_{V,U}) \circ (c_{V,W} \otimes \mathrm{Id}_U),$$

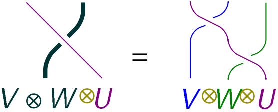
$$c_{V\otimes W,U} = (c_{V,U} \otimes \mathrm{Id}_W) \circ (\mathrm{Id}_V \otimes c_{W,U}).$$





Braided categories



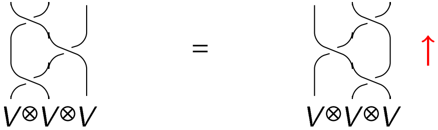


Braided objects

Definition

A *braided object* in $\mathscr{C}$ is an object V endowed with a *braiding*, i.e. a morphism $\sigma_V : V \otimes V \to V \otimes V$ satisfying (a categorical version of) the *Yang-Baxter equation*:

$$(\sigma_V \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_V \otimes \sigma_V) \circ (\sigma_V \otimes \mathrm{Id}_V) = (Id_V \otimes \sigma_V) \circ (\sigma_V \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_V \otimes \sigma_V).$$



Yang-Baxter equation ↔ Reidemeister move III

Braided objects

Definition

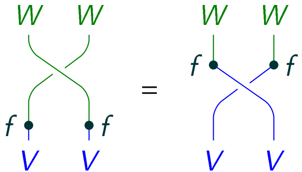
A *braided object* in $\mathscr{C}$ is an object V endowed with a *braiding*, i.e. a morphism $\sigma_V : V \otimes V \to V \otimes V$ satisfying the *Yang-Baxter equation*:

$$(\sigma_V \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_V \otimes \sigma_V) \circ (\sigma_V \otimes \mathrm{Id}_V) = (Id_V \otimes \sigma_V) \circ (\sigma_V \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_V \otimes \sigma_V).$$

A *braided morphism* is a morphism $f : (V, \sigma_V) \to (W, \sigma_W)$ respecting the braidings:

$$(f \otimes f) \circ \sigma_V = \sigma_W \circ (f \otimes f) : V \otimes V \to W \otimes W.$$

$\rightsquigarrow$ a category $\mathbf{Br}(\mathscr{C})$.



Braided categories vs. braided objects

braided categories	braided objects
"global" notion	"local" notion

Any object in a braided category is braided.

Remark

We should actually talk about *weakly braided* or *pre-braided* categories / objects, since we do not demand $c_{V,W}$ (or σ_V) to be invertible.

Braided categories vs. braided objects: a digression

Theorem (Folklore)

Denote by $\mathscr{C}_{gl-br}$ the free braided category generated by a single object V. Then for each n one has a monoid isomorphism

$$\boxed{\mathrm{End}_{\mathscr{C}_{gl-br}}(V^{\otimes n}) \xrightarrow{\sim} B_n^+}$$

$$\mathrm{Id}_{i-1} \otimes c_{V,V} \otimes \mathrm{Id}_{n-i-1} \longmapsto \sigma_i.$$

Here B_n^+ is the *positive Artin braid monoid*:

- algebraically: generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$, subject to relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \text{if } |i-j| > 1, 1 \leqslant i,j \leqslant n-1, \qquad (Br_C)$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \qquad \forall\, 1 \leqslant i \leqslant n-2; \qquad (Br_{YB})$$

- topologically: braids with positive crossings only.

Braided categories vs. braided objects: a digression

Theorem (Folklore)

Denote by $\mathscr{C}_{loc-br}$ the free monoidal category generated by a single braided object (V,σ_V). Then for each n one has a monoid isomorphism

$$\boxed{\mathrm{End}_{\mathscr{C}_{loc-br}}(V^{\otimes n}) \xrightarrow{\sim} B_n^+}$$

$$\mathrm{Id}_{i-1} \otimes \sigma_V \otimes \mathrm{Id}_{n-i-1} \longmapsto \sigma_i.$$

Here B_n^+ is the *positive Artin braid monoid*:

- algebraically: generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$, subject to relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad \text{if } |i-j|>1, 1 \leqslant i,j \leqslant n-1, \qquad (Br_C)$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad \forall\, 1 \leqslant i \leqslant n-2; \qquad (Br_{YB})$$

- topologically: braids with positive crossings only.

Braided categories vs. braided objects: a digression

Theorem (L., 2012)

Denote by $\mathscr{C}_{loc-gl-br}$ the free symmetric category generated by a single braided object (V,σ_V). Then for each n one has a monoid isomorphism

$$\operatorname{End}_{\mathscr{C}_{loc-gl-br}}(V^{\otimes n}) \xrightarrow{\sim} VB_n^+$$

$$\mathrm{Id}_{i-1} \otimes c_{V,V} \otimes \mathrm{Id}_{n-i-1} \longmapsto \zeta_i,$$

$$\mathrm{Id}_{i-1} \otimes \sigma_V \otimes \mathrm{Id}_{n-i-1} \longmapsto \sigma_i.$$

Here VB_n^+ is the *positive virtual braid monoid* (Kauffman, Vershinin):

- algebraically: generators $\{\sigma_i, \zeta_i, 1 \leqslant i \leqslant n-1\}$, subject to
  - ✓(Br_C) and (Br_{YB}) for the σ_i's;
  - ✓(Br_C) and (Br_{YB}) for the ζ_i's; $\quad \zeta_i\zeta_i = 1 \quad \forall i$;
  - ✓ $\left.\begin{aligned} \sigma_i\zeta_j = \zeta_j\sigma_i \quad & \forall |i-j|>1, \\ \zeta_i\zeta_{i+1}\sigma_i = \sigma_{i+1}\zeta_i\zeta_{i+1} \quad & \forall i \end{aligned}\right\}$ mixed relations.

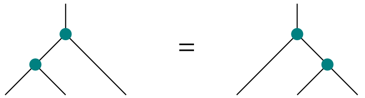
Unital associative algebras

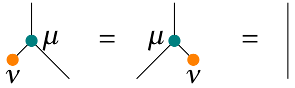
Definition

A *unital associative algebra* (= UAA) in $\mathscr{C}$ is an object V together with morphisms $\mu : V \otimes V \to V$ and $\nu : \mathbf{I} \to V$, satisfying the associativity and the unit conditions:

$$\mu \circ (\mu \otimes \mathrm{Id}_V) = \mu \circ (\mathrm{Id}_V \otimes \mu) : V^3 \to V,$$
$$\mu \circ (\nu \otimes \mathrm{Id}_V) = \mu \circ (\mathrm{Id}_V \otimes \nu) = \mathrm{Id}_V .$$

$\rightsquigarrow$ category $\mathbf{Alg}(\mathscr{C})$.





UAAs as braided objects

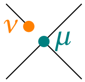
Theorem (L., 2012)

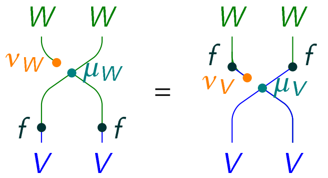
One has a functor

$$\boxed{\mathbf{Alg}(\mathscr{C}) \longrightarrow \mathbf{Br}(\mathscr{C})}$$

$$(V, \mu, \nu) \longmapsto (V, \sigma_{Ass} = \nu \otimes \mu),$$

$$(f : V \to W) \longmapsto (f : V \to W).$$





UAAs as braided objects

Definition

Denote by $\mathbf{Br_\bullet}(\mathscr{C})$ the category of *pointed braided objects* in $\mathscr{C}$:

- objects : braided objects V endowed with a "unit" $\nu : \mathbf{I} \to V$;
- morphisms : braided morphisms in $\mathscr{C}$ respecting units, i.e. $f \circ \nu_V = \nu_W$ for $f : V \to W$.

A better theorem (L., 2012)

One has a fully faithful functor

$$\mathbf{Alg}(\mathscr{C}) \longrightarrow \mathbf{Br_\bullet}(\mathscr{C})$$

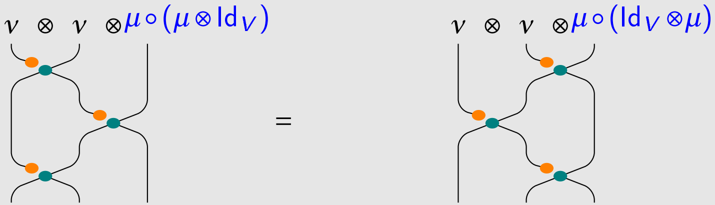
$$(V, \mu, \nu) \longmapsto (V, \sigma_{Ass}, \nu),$$

$$(f : V \to W) \longmapsto (f : V \to W).$$

UAAs as braided objects

Remarks

- Yang-Baxter for σ_{Ass} $\iff$ associativity for μ.



- $\sigma_{Ass}\circ\sigma_{Ass} = \sigma_{Ass}$ $\Longrightarrow$ highly non-invertible.
- $\sigma = \nu\otimes\mu + \mu\otimes\nu - \mathrm{Id}$ also encodes the associativity (Nuss, Nichita).
- Dual picture: $\mathbf{coAlg}(\mathscr{C}) \hookrightarrow \mathbf{Br}^{\bullet}(\mathscr{C})$

$$\mathbf{coAlg}(\mathscr{C}) \hookrightarrow \mathbf{Br}^{\bullet}_{\bullet}(\mathscr{C}) \hookleftarrow \mathbf{Alg}(\mathscr{C})$$

Unital Leinbiz algebras

Definition

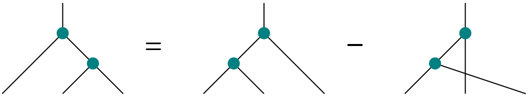
A *unital Leinbiz algebra* (= ULA) in a symmetric preadditive category $\mathscr{C}$ is an object V together with morphisms $[,]: V \otimes V \to V$ and $\nu : \mathbf{I} \to V$, satisfying the Leinbiz and the Lie unit conditions:

$$[,]\circ(\mathrm{Id}_V \otimes [,]) = [,]\circ([,]\otimes \mathrm{Id}_V) - [,]\circ([,]\otimes \mathrm{Id}_V)\circ(\mathrm{Id}_V \otimes c_{V,V}) : V^{\otimes 3} \to V,$$

$$[,]\circ(\mathrm{Id}_V \otimes \nu) = [,]\circ(\nu \otimes \mathrm{Id}_V) = 0 : V \to V.$$

$\rightsquigarrow$ category $\mathbf{Lei}(\mathscr{C})$.

A non-commutative version of Lie algebras (Loday, Cuvier).



ULAs as braided objects

Theorem (L., 2012)

One has a fully faithful functor

$$\boxed{\mathbf{Lei}(\mathscr{C}) \hookrightarrow \mathbf{Br}_{\bullet}(\mathscr{C})}$$

$$(V,[,],\nu) \longmapsto (V, \sigma_{Lei} = \nu \otimes [,] + c_{V,V}, \nu),$$

$$(f : V \to W) \longmapsto (f : V \to W).$$

ULAs as braided objects

Theorem (L., 2012)

$$\mathbf{Lei}(\mathscr{C}) \hookrightarrow \mathbf{Br_{\bullet}}(\mathscr{C})$$

$$(V,[,],\nu) \longmapsto (V, \sigma_{Lei} = \nu \otimes [,] + c_{V,V}, \nu).$$

Remarks

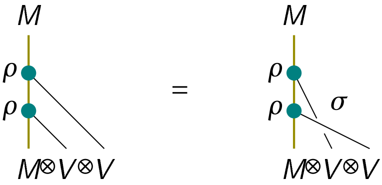
- Yang-Baxter for σ_{Lei} $\iff$ Leibniz condition for $[,]$.
- A conceptual explication of the choice of the lift of the Jacobi condition.
- σ_{Lei} was previously considered for Lie algebras.
- σ_{Lei} is invertible.
- Dual picture: co-Leibniz algebras.

Braided modules : definition

Definition

A *right braided module* over a braided object (V,σ) in $\mathscr{C}$ is an object $M \in \mathrm{Ob}(\mathscr{C})$ equipped with a morphism $\rho : M \otimes V \to M$ satisfying

$$\rho \circ (\rho \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_M \otimes \sigma) = \rho \circ (\rho \otimes \mathrm{Id}_V) : M \otimes V \otimes V \to M.$$



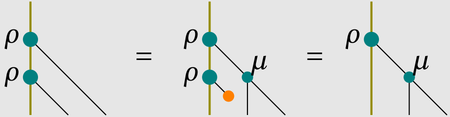
Braided modules : examples

All braided modules are supposed *normalized* here, i.e. $\rho \circ (\mathrm{Id}_M \otimes \nu) = \mathrm{Id}_M$.

Examples

- UAAs: usual modules over associative algebras

$$\rho \circ (\rho \otimes \mathrm{Id}_V) = \rho \circ (\mathrm{Id}_M \otimes \mu)$$



- ULAs: usual Leibniz modules

$$\rho \circ (\rho \otimes \mathrm{Id}_V) = \rho \circ (\rho \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_M \otimes c_{V,V}) + \rho \circ (\mathrm{Id}_M \otimes [,])$$

Braided homology

Theorem (L.,2012)

In a preadditive monoidal category $\mathscr{C}$, *take*

✓*a braided object* (V,σ)*;*

✓*a right and a left braided* V*-modules* (M,ρ) *and* (N,λ)*.*

Then the morphisms

$$^{\rho}d_n := \sum_{i=1}^{n} (-1)^{i-1}\, {^{\rho}d_{n;i}}, \qquad d_n^{\lambda} := \sum_{i=1}^{n} (-1)^{i-1}\, d_{n;i}^{\lambda},$$

$$^{\rho}d_{n;i} := (\rho \otimes \mathrm{Id}_V^{n-1} \otimes \mathrm{Id}_N) \circ (\mathrm{Id}_M \otimes (\sigma_2 \circ \sigma_3 \circ \cdots \circ \sigma_i) \otimes \mathrm{Id}_N)$$

$$d_{n;i}^{\lambda} := (\mathrm{Id}_M \otimes \mathrm{Id}_V^{n-1} \otimes \lambda) \circ (\mathrm{Id}_M \otimes (\sigma_n \circ \cdots \circ \sigma_{i+1}) \otimes \mathrm{Id}_N)$$

define a bidegree -1 *tensor bidifferential on* $M \otimes V^{\otimes n} \otimes N$*.*
(Here $\sigma_j = \mathrm{Id}_M \otimes \mathrm{Id}_V^{j-2} \otimes \sigma \otimes \mathrm{Id}_V^{n-j} \otimes \mathrm{Id}_N$*.)*

Braided homology

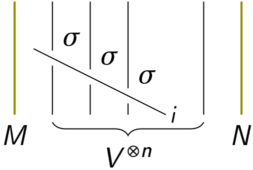
Theorem (L.,2012)

$$ {}^{\rho}d_n := \sum_{i=1}^{n} (-1)^{i-1}\, {}^{\rho}d_{n;i}, \qquad d_n^{\lambda} := \sum_{i=1}^{n} (-1)^{i-1}\, d_{n;i}^{\lambda}, $$

$$ {}^{\rho}d_{n;i} := (\rho \otimes \mathrm{Id}_V^{n-1} \otimes \mathrm{Id}_N) \circ (\mathrm{Id}_M \otimes (\sigma_2 \circ \sigma_3 \circ \cdots \circ \sigma_i) \otimes \mathrm{Id}_N) $$

$$ d_{n;i}^{\lambda} := (\mathrm{Id}_M \otimes \mathrm{Id}_V^{n-1} \otimes \lambda) \circ (\mathrm{Id}_M \otimes (\sigma_n \circ \cdots \circ \sigma_{i+1}) \otimes \mathrm{Id}_N) $$

(Here $\sigma_j = \mathrm{Id}_M \otimes \mathrm{Id}_V^{i-2} \otimes \sigma \otimes \mathrm{Id}_V^{n-i} \otimes \mathrm{Id}_N$.)

${}^{\rho}d_{n;i} =$ 

Braided homology

Theorem (L.,2012)

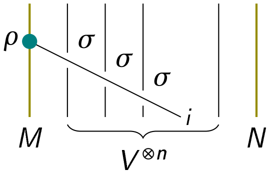
$$ {}^{\rho}d_n := \sum_{i=1}^{n} (-1)^{i-1}\, {}^{\rho}d_{n;i}, \qquad d_n^{\lambda} := \sum_{i=1}^{n} (-1)^{i-1}\, d_{n;i}^{\lambda}, $$

$$ {}^{\rho}d_{n;i} := (\rho \otimes \mathrm{Id}_V^{n-1} \otimes \mathrm{Id}_N) \circ (\mathrm{Id}_M \otimes (\sigma_2 \circ \sigma_3 \circ \cdots \circ \sigma_i) \otimes \mathrm{Id}_N) $$

$$ d_{n;i}^{\lambda} := (\mathrm{Id}_M \otimes \mathrm{Id}_V^{n-1} \otimes \lambda) \circ (\mathrm{Id}_M \otimes (\sigma_n \circ \cdots \circ \sigma_{i+1}) \otimes \mathrm{Id}_N) $$

(Here $\sigma_j = \mathrm{Id}_M \otimes \mathrm{Id}_V^{i-2} \otimes \sigma \otimes \mathrm{Id}_V^{n-i} \otimes \mathrm{Id}_N$.)

${}^{\rho}d_{n;i} =$



Braided homology

Remarks

- A family of differentials $\rightsquigarrow$ linear combinations.
- A *pre-bisimplicial* (or *pre-cubical*) structure on $M \otimes V^{\otimes n} \otimes N$, which can be upgraded into a *weakly bisimplicial* one (a "nice" comultiplication on $V \rightsquigarrow$ degeneracies).
- The construction is *functorial*.
- The differentials can be interpreted in terms of *quantum shuffles* (Rosso).
- A generalization: *Loday's hyperboundaries* $M \otimes V^{\otimes n} \otimes N \to M \otimes V^{\otimes n-k} \otimes N$.
- Interesting homology morphisms.
- Duality: a cohomology version.

Braided homology: examples

Examples

- UAA V + algebra module M $\rightsquigarrow$
braided object (V, σ_{Ass}) + braided module M $\rightsquigarrow$
bar / Hochschild complex
- ULA V + Leibniz module M $\rightsquigarrow$
braided object (V, σ_{Lei}) + braided module M $\rightsquigarrow$
Leibniz (=non-commutative Chevalley-Eilenberg) complex

algebraic structure $\rightsquigarrow$ chain complex

Braided homology: examples

Examples

- UAA V + algebra module $M \rightsquigarrow$
braided object (V, σ_{Ass}) + braided module $M \rightsquigarrow$
bar / Hochschild complex

- ULA V + Leibniz module $M \rightsquigarrow$
braided object (V, σ_{Lei}) + braided module $M \rightsquigarrow$
Leibniz (=non-commutative Chevalley-Eilenberg) complex

algebraic structure $\overset{\text{case by case}}{\rightsquigarrow}$ **braiding** $\overset{\text{Theorem}}{\longmapsto}$ chain complex

Braided systems: definition

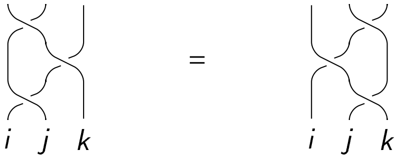
Definition

A *braided system* in $\mathscr{C}$ is a family $V_1, V_2, \ldots, V_r \in \mathrm{Ob}(\mathscr{C})$ endowed with a *multi-braiding*, i.e. morphisms $\sigma_{i,j} : V_i \otimes V_j \to V_j \otimes V_i \ \forall \underline{i \leqslant j}$, satisfying the multi-Yang-Baxter equation

$$(\sigma_{j,k} \otimes \mathrm{Id}_i) \circ (\mathrm{Id}_j \otimes \sigma_{i,k}) \circ (\sigma_{i,j} \otimes \mathrm{Id}_k) = (\mathrm{Id}_k \otimes \sigma_{i,j}) \circ (\sigma_{i,k} \otimes \mathrm{Id}_j) \circ (\mathrm{Id}_i \otimes \sigma_{j,k})$$

on all the tensor products $V_i \otimes V_j \otimes V_k$ with $\underline{i \leqslant j \leqslant k}$.

$\rightsquigarrow$ category ${}_r\mathbf{BrSyst}(\mathscr{C})$.



Braided systems: representations and homology

Definition

A *braided system* in $\mathscr{C}$ is a family $V_1, V_2, \ldots, V_r \in \mathrm{Ob}(\mathscr{C})$ endowed with a *multi-braiding* $\sigma_{i,j} : V_i \otimes V_j \to V_j \otimes V_i \ \forall\, i \leqslant j$, satisfying YBE on all the tensor products $V_i \otimes V_j \otimes V_k$ with $i \leqslant j \leqslant k$.

A *multi-braided module* over a $(\overline{V}, \overline{\sigma}) \in {}_r\mathbf{BrSyst}(\mathscr{C})$ is an $M \in \mathrm{Ob}(\mathscr{C})$ equipped with $(\rho_i : M \otimes V_i \to M)_{1 \leqslant i \leqslant r}$ satisfying $\forall\, i \leqslant j$

$$\rho_j \circ (\rho_i \otimes \mathrm{Id}_j) = \rho_i \circ (\rho_j \otimes \mathrm{Id}_i) \circ (\mathrm{Id}_M \otimes \sigma_{i,j}) : M \otimes V_i \otimes V_j \to M.$$

Theorem (L.,2012)

A bidifferential structure on $M \otimes T(\overline{V})_n^{\rightarrow} \otimes N$, where $T(\overline{V})_n^{\rightarrow}$ is the direct sum of all the tensor products of type

$$V_1^{\otimes m_1} \otimes V_2^{\otimes m_2} \otimes \cdots \otimes V_r^{\otimes m_r}, \qquad m_i \geqslant 0, \sum m_i = n.$$

Example: bialgebras as a braided system

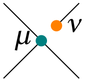
Theorem (L.,2012)

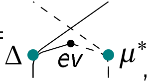
The groupoid $^*\mathbf{Bialg}(\mathbf{vect}_{\Bbbk})$ *of bialgebras and bialgebra* <u>*iso*</u>*morphisms in* $\mathbf{vect}_{\Bbbk}$ *is a subcategory of the groupoid of size* 2 *bipointed braided systems:*

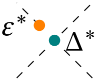
$$^*\mathbf{Bialg} \hookrightarrow {}^*_2\mathbf{BrSyst}^{\bullet}_{\bullet}$$

$$(H,\mu,\nu,\Delta,\varepsilon) \longmapsto \overline{H}_{bi} := (V_1 := H, V_2 := H^*; \qquad \nu,\varepsilon^*;\varepsilon,\nu^*;$$
$$\sigma_{1,1} := \sigma^r_{Ass}(H), \sigma_{2,2} := \sigma_{Ass}(H^*), \sigma_{1,2} = \sigma_{bi}),$$
$$f \longmapsto (f,(f^{-1})^*),$$

where $\sigma_{bi}(h \otimes l) := \langle l_{(1)}, h_{(2)} \rangle\, l_{(2)} \otimes h_{(1)}$.

$\sigma_{H,H} =$ 

, $\sigma_{H,H^*} =$ 

, $\sigma_{H^*,H^*} =$ 

Example: bialgebras as a braided systems

Theorem (continued)

- *YBE on* $H \otimes H \otimes H^*$ $\iff$ *the bialgebra compatibility condition.*
- *Invertibility of* σ_{bi} $\iff$ *H is a Hopf algebra.*
- *Braided modules for* $\overline{H}_{bi}$ $\simeq$ *right-right Hopf modules over H.*
- *Braided homology for* $\overline{H}_{bi}$ $\supseteq$ *Gerstenhaber-Schack homology for H.*

Summary: "braided" interpretation for algebraic structures

(multi-)braiding	$\leftarrow\!\shortmid$	algebraic structure
$_r$**BrSyst**$(\mathscr{C})$	$\hookleftarrow$	**Struc**$(\mathscr{C})$
YBE	$\Leftrightarrow$	the defining relation
invertibility	$\Leftrightarrow$	algebraic properties
braided morphisms	$\simeq$	structural morphisms
braided modules	$\simeq$	usual modules
braided differentials	$\supseteq$	usual differentials

Examples

- ⇝ UAAs
- ⇝ ULAs
- ⇝ bialgebras
- ⇝ crossed / smash products
- ⇝ self-distributive structures
- ⇝ Yetter-Drinfel′d modules

"algebraic structure = braiding"

Thank you!