Laver tables: topological applications of set-theoretic constructions

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Joint work with **Patrick DEHORNOY** (University of Caen)

Knots and Low Dimensional Manifolds
Busan, Korea, August 25, 2014

$0, 1, 2, 3, \ldots;$
$\aleph_0, \aleph_1, \aleph_2, \ldots;$
$\aleph_\omega, \ldots$

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
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6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
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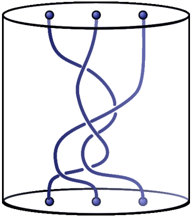

1 A Laver table is...

Basic definitions

A shelf (= self-distributive structure)

is a set S with an operation $\triangleright$ satisfying

$$\boxed{a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright (a \triangleright c)} \qquad \text{(SD)}$$

Example: group G, $f \triangleright g = fgf^{-1}$.

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is a free shelf generated by a single element γ.

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Theorem (Laver, '95): properties (SD) and (Init) uniquely define $\triangleright$.

⚠ False for $\{1, 2, 3, \ldots, q\}$ with $q \neq 2^n$.

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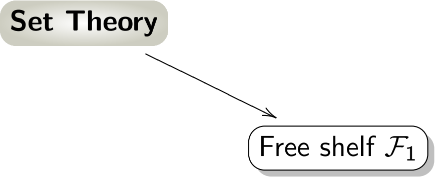
$$\gamma = 1 \qquad (\gamma \triangleright \gamma) \triangleright \gamma = 3$$

$$\gamma \triangleright \gamma = 2 \qquad ((\gamma \triangleright \gamma) \triangleright \gamma) \triangleright \gamma = 4 \qquad \ldots$$

Laver tables in Set Theory



Richard Laver

- $\mathcal{F}_1$ is realized inside the self-embedding shelf of a large cardinal:

$$\mathcal{F}_1 \cong F \subseteq \mathrm{Emb}(V_\lambda).$$

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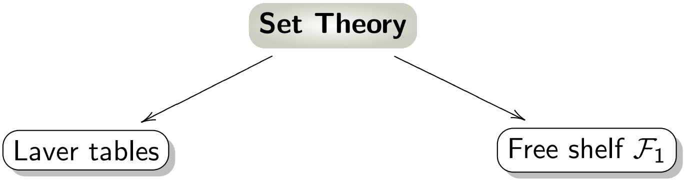

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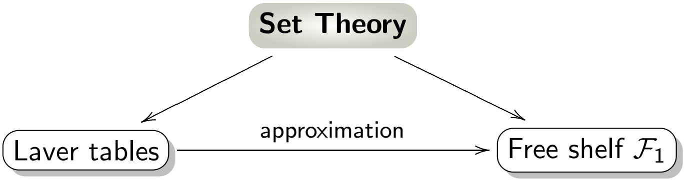

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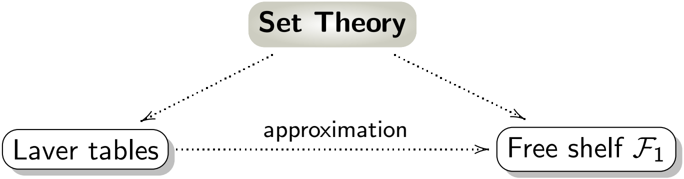

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⚠ Everything works only under an unprovable set-theoretic axiom.

Going beyond Set Theory?

Elementary definition

$A_n = (\{\,1, 2, 3, \ldots, 2^n\,\}, \triangleright)$ satisfying

$$a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright (a \triangleright c) \qquad \& \qquad a \triangleright 1 \equiv a + 1 \mod 2^n.$$

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Elementary properties

❀ A projective system of shelves.

A_1	1	2
1	2	2
2	1	2

A_2	1	2	3	4
1	2	4	2	4
2	3	4	3	4
3	4	4	4	4
4	1	2	3	4

$\xrightarrow{\text{mod } 2}$

A_2	1	2	1	2
1	2	2	2	2
2	1	2	1	2
1	2	2	2	2
2	1	2	1	2

Going beyond Set Theory?

Elementary properties

- A projective system of shelves.
- Periodic rows.

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
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$\pi_3(1) = 4$
$\pi_3(2) = 4$
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$\pi_3(6) = 2$
$\pi_3(7) = 1$
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Going beyond Set Theory?

Elementary properties

- A projective system of shelves.
- Periodic rows.
- Solutions of $p \triangleright q = q$.
- Some "nice" rows and columns.

A_3	1	2	3	4	5	6	7	8	
				8				8	
				8				8	
				8				8	
4	5	6	7	8	5	6	7	8	$\pi_3(4) = 4$
				8				8	
				8				8	
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  ⚠ No closed formulas for $p \triangleright q$, nor for $\pi_n(p)$.

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1	2	4	6	8	2	4	6	8	$\pi_n(1) = ?$
				8				8	
				8				8	
4	5	6	7	8	5	6	7	8	$\pi_3(4) = 4$
				8				8	
				8				8	
7	8	8	8	8	8	8	8	8	$\pi_3(7) = 1$
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- $\pi_n(1) \underset{n \to \infty}{\longrightarrow} \infty$.

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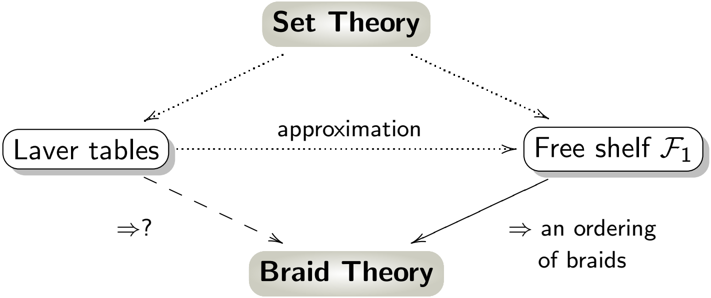
⚠ Theorems under Axiom I3!

A_4	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	2	12	14	16	2	12	14	16	2	12	14	16	2	12	14	16
2	3	12	15	16	3	12	15	16	3	12	15	16	3	12	15	16
3	4	8	12	16	4	8	12	16	4	8	12	16	4	8	12	16
4	5	6	7	8	13	14	15	16	5	6	7	8	13	14	15	16
5	6	8	14	16	6	8	14	16	6	8	14	16	6	8	14	16
6	7	8	15	16	7	8	15	16	7	8	15	16	7	8	15	16
7	8	16	8	16	8	16	8	16	8	16	8	16	8	16	8	16
8	9	10	11	12	13	14	15	16	9	10	11	12	13	14	15	16
9	10	12	14	16	10	12	14	16	10	12	14	16	10	12	14	16
10	11	12	15	16	11	12	15	16	11	12	15	16	11	12	15	16
11	12	16	12	16	12	16	12	16	12	16	12	16	12	16	12	16
12	13	14	15	16	13	14	15	16	13	14	15	16	13	14	15	16
13	14	16	14	16	14	16	14	16	14	16	14	16	14	16	14	16
14	15	16	15	16	15	16	15	16	15	16	15	16	15	16	15	16
15	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
16	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

⚠ Rich combinatorics.

2 Dreams: braid invariants

Laver tables in Topology

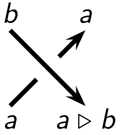


Richard Laver

Patrick Dehornoy

Shelf colorings

Colorings by $(S, \triangleright)$:



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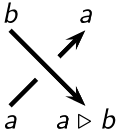

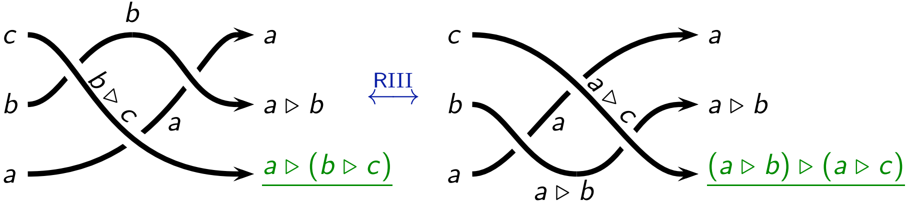

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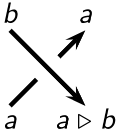

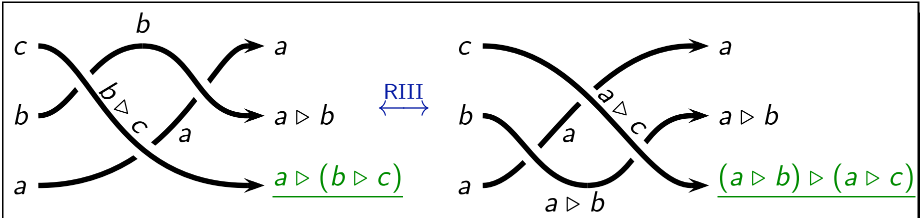

RIII $\leftrightarrow$ $a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright (a \triangleright c)$ (SD)

positive braid invariants $\overset{\text{colorings}}{\leftsquigarrow}$ shelf

Shelf colorings

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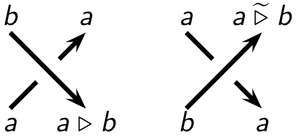

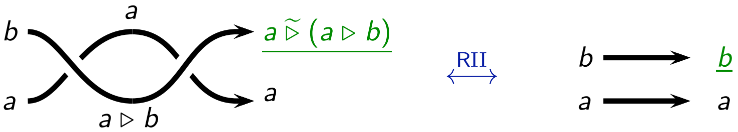

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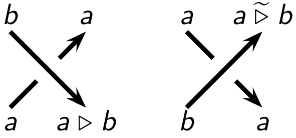

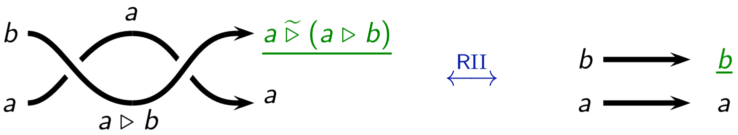

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} **Rack**

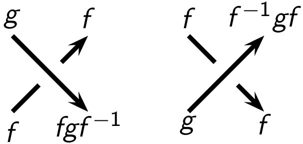
braid invariants $\overset{\text{colorings}}{\leftsquigarrow}$ rack

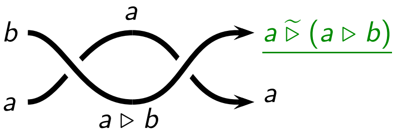
Shelf colorings

Colorings by G

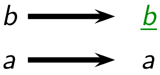
$\updownarrow$

$Rep(\pi_1((\mathbb{R}^2 \times [0,1])\backslash\beta), G)$





$\overset{\text{RII}}{\longleftrightarrow}$



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} **Rack**

braid invariants $\overset{\text{colorings}}{\leftsquigarrow}$ rack

Example: Group $G \rightsquigarrow$ a rack $(G\,,\ f \triangleright g = fgf^{-1}\,,\ f \,\widetilde{\triangleright}\, g = f^{-1}gf)$.

$\mathcal{F}_1$-colorings for arbitrary braids

positive braid invariants $\overset{\text{colorings}}{\leftsquigarrow}$ $\mathcal{F}_1$ or A_n

Question: What about arbitrary braid invariants?

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Problem: $\mathcal{F}_1$ and the A_n are shelves, but not racks.

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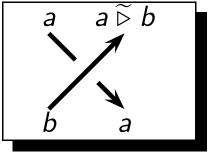

$$\text{RII} \quad \leftrightarrow \quad a \mathbin{\widetilde{\triangleright}} (a \triangleright b) = b = a \triangleright (a \mathbin{\widetilde{\triangleright}} b)$$

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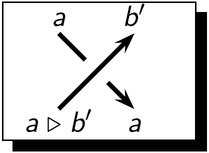

RII $\leftrightarrow$ maps $T_a : b \mapsto a \triangleright b$ are invertible

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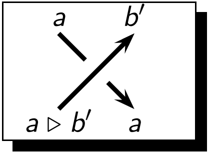

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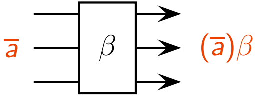

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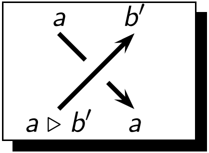

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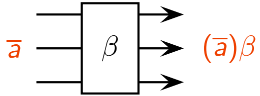

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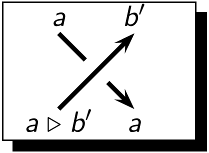

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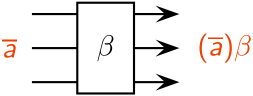

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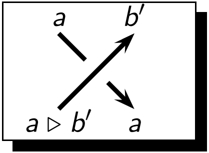

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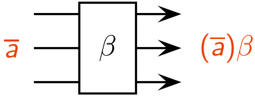

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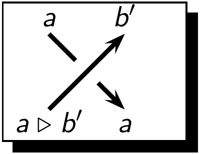

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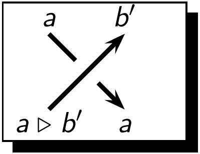

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Why do we persist?

- Conjecturally, $A_n \underset{n\to\infty}{\to} A_\infty \supseteq \mathcal{F}_1$.
- A_n are finite.

3 Reality: 2- and 3-cocycles

Shelf colorings revisited

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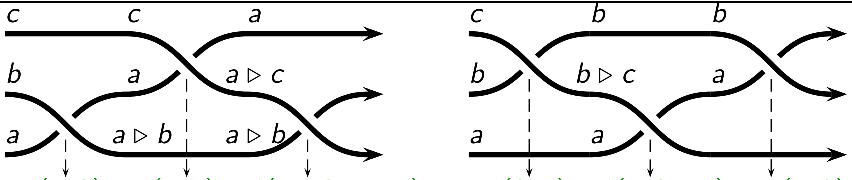

$\cancel{\phi(a, b)} + \phi(a, c) + \phi(a \triangleright b, a \triangleright c)$ $\qquad$ $\phi(b, c) + \phi(a, b \triangleright c) + \cancel{\phi(a, b)}$

A variation: shadow colorings

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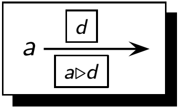

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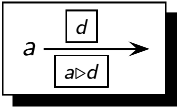

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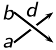

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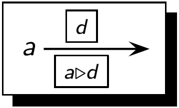

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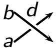

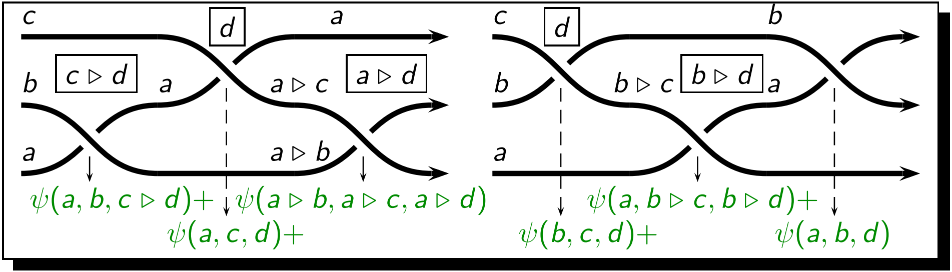

Weights via cocycles

Rack cohomology (Fenn-Rourke-Sanderson, '95)

Shelf $(S, \triangleright) \rightsquigarrow$ complex $(\mathrm{Hom}(S^{\times k}, \mathbb{Z}), d_{\mathrm{R}}^k) \rightsquigarrow H_{\mathrm{R}}^k(S)$

$$(d_{\mathrm{R}}^k f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} (f(a_1, \ldots, a_{i-1}, a_i \triangleright a_{i+1}, \ldots, a_i \triangleright a_{k+1}) - f(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{k+1})).$$

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2- and 3-cocycles for Laver tables

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Remark: 2-cocycles capture the combinatorics of the A_n (e.g., periods).

$\phi_{1,3}$	1
1	1
2	1
3	1
4	1
5	1
6	1
7	1
8	·

$\phi_{2,3}$	1	2	3	4	5	6	7	8
1	·	1	·	·	·	·	·	·
2	1	1	·	·	1	·	·	·
3	1	1	·	·	1	·	·	·
4	·	1	·	·	·	·	·	·
5	1	1	·	·	1	·	·	·
6	1	1	·	·	1	·	·	·
7	1	1	·	·	1	·	·	·
8	·	·	·	·	·	·	·	·

$\phi_{3,3}$	1	2	3	4	5	6	7	8
1	1	·	1	·	1	·	·	·
2	·	·	1	·	·	·	·	·
3	1	·	1	·	1	·	·	·
4	·	·	1	·	·	·	·	·
5	1	·	1	·	1	·	·	·
6	1	·	1	·	1	·	·	·
7	1	·	1	·	1	·	·	·
8	·	·	·	·	·	·	·	·

$\phi_{4,3}$	1	2	3	4	5	6	7	8
1	·	·	·	1	·	·	·	·
2	·	·	·	1	·	·	·	·
3	·	1	·	1	·	1	·	·
4	·	·	·	1	·	·	·	·
5	·	1	·	1	·	1	·	·
6	·	1	·	1	·	1	·	·
7	1	1	1	1	1	1	1	·
8	·	·	·	·	·	·	·	·

$\phi_{5,3}$	1	2	3	4	5	6	7	8
1	1	·	·	·	1	·	·	·
2	1	·	·	·	1	·	·	·
3	1	·	·	·	1	·	·	·
4	·	·	·	·	·	·	·	·
5	1	·	·	·	1	·	·	·
6	1	·	·	·	1	·	·	·
7	1	·	·	·	1	·	·	·
8	·	·	·	·	·	·	·	·

$\phi_{6,3}$	1	2	3	4	5	6	7	8
1	·	1	·	·	·	1	·	·
2	·	1	·	·	·	1	·	·
3	1	1	1	·	1	1	1	·
4	·	·	·	·	·	·	·	·
5	·	1	·	·	·	1	·	·
6	·	1	·	·	·	1	·	·
7	1	1	1	·	1	1	1	·
8	·	·	·	·	·	·	·	·

$\phi_{7,3}$	1	2	3	4	5	6	7	8
1	1	·	1	·	1	·	1	·
2	·	·	·	·	·	·	·	·
3	1	·	1	·	1	·	1	·
4	·	·	·	·	·	·	·	·
5	1	·	1	·	1	·	1	·
6	·	·	·	·	·	·	·	·
7	1	·	1	·	1	·	1	·
8	·	·	·	·	·	·	·	·

$\phi_{1,3}$	1
1	1
2	1
3	1
4	1
5	1
6	1
7	1
8	·

$\phi_{2,3}$	1	2	3	4	5	6	7	8
1	·	1	·	·	·	·	·	·
2	1	1	·	·	1	·	·	·
3	1	1	·	·	1	·	·	·
4	·	1	·	·	·	·	·	·
5	1	1	·	·	1	·	·	·
6	1	1	·	·	1	·	·	·
7	1	1	·	·	1	·	·	·
8	·	·	·	·	·	·	·	·

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1	1	·	1	·	1	·	·	·
2	·	·	1	·	·	·	·	·
3	1	·	1	·	1	·	·	·
4	·	·	1	·	·	·	·	·
5	1	·	1	·	1	·	·	·
6	1	·	1	·	1	·	·	·
7	1	·	1	·	1	·	·	·
8	·	·	·	·	·	·	·	·

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1	·	·	·	1	·	·	·	·
2	·	·	·	1	·	·	·	·
3	·	1	·	1	·	1	·	·
4	·	·	·	1	·	·	·	·
5	·	1	·	1	·	1	·	·
6	·	1	·	1	·	1	·	·
7	1	1	1	1	1	1	1	·
8	·	·	·	·	·	·	·	·

$\phi_{5,3}$	1	2	3	4	5	6	7	8
1	1	·	·	·	1	·	·	·
2	1	·	·	·	1	·	·	·
3	1	·	·	·	1	·	·	·
4	·	·	·	·	·	·	·	·
5	1	·	·	·	1	·	·	·
6	1	·	·	·	1	·	·	·
7	1	·	·	·	1	·	·	·
8	·	·	·	·	·	·	·	·

$\phi_{6,3}$	1	2	3	4	5	6	7	8
1	·	1	·	·	·	1	·	·
2	·	1	·	·	·	1	·	·
3	1	1	1	·	1	1	1	·
4	·	·	·	·	·	·	·	·
5	·	1	·	·	·	1	·	·
6	·	1	·	·	·	1	·	·
7	1	1	1	·	1	1	1	·
8	·	·	·	·	·	·	·	·

$\phi_{7,3}$	1	2	3	4	5	6	7	8
1	1	·	1	·	1	·	1	·
2	·	·	·	·	·	·	·	·
3	1	·	1	·	1	·	1	·
4	·	·	·	·	·	·	·	·
5	1	·	1	·	1	·	1	·
6	·	·	·	·	·	·	·	·
7	1	·	1	·	1	·	1	·
8	·	·	·	·	·	·	·	·

Right division for Laver tables

Important proof ingredient: **right division** relation

$$\boxed{a \,|_r\, b \quad \Longleftrightarrow \quad b = c \triangleright a \text{ for some } c}$$

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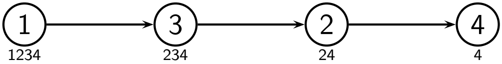
Properties:

- Minimal element: 1.
- Maximal element: 2^n.
- Not linear for $n \geqslant 3$.
- Not lattice for $n \geqslant 5$.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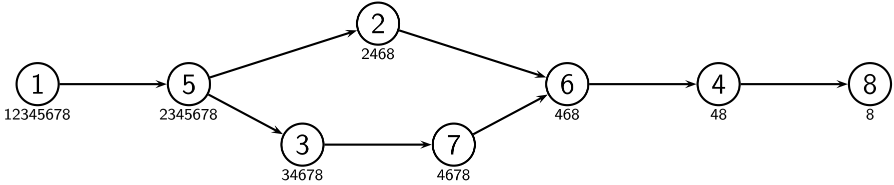
3
2
4
1234
234
24
4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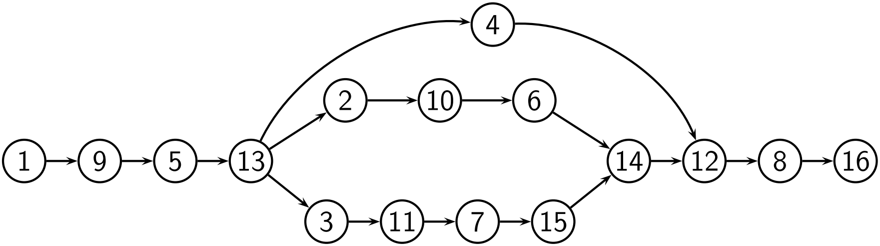
4
2
10
6
1
9
5
13
14
12
8
16
3
11
7
15

Digression: Laver tables and branched braids

Theorem (Laver, Drápal, 95)

Operation $\boxed{p \circ q = p \triangleright (q+1) - 1}$ satisfies

$$(a \circ b) \triangleright c = a \triangleright (b \triangleright c), \qquad (a \circ b) \circ c = a \circ (b \circ c),$$

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$$a \circ b = (a \triangleright b) \circ a, \qquad a \circ 2^n = a.$$

$A_3, \circ$	1	2	3	4	5	6	7	8
1	3	5	7	1	3	5	7	1
2	3	6	7	2	3	6	7	2
3	7	3	7	3	7	3	7	3
4	5	6	7	4	5	6	7	4
5	7	5	7	5	7	5	7	5
6	7	6	7	6	7	6	7	6
7	7	7	7	7	7	7	7	7
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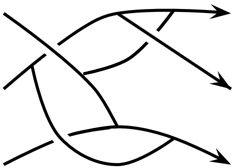
Application: colorings of positive **branched braids**.



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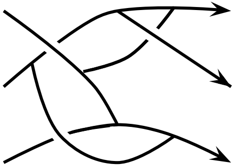

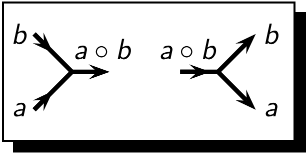

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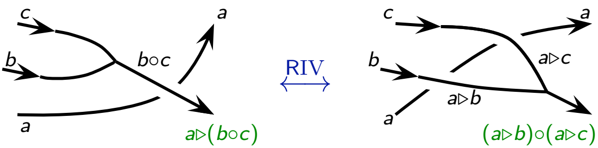

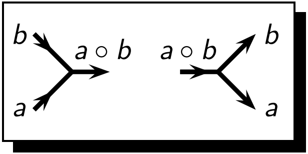

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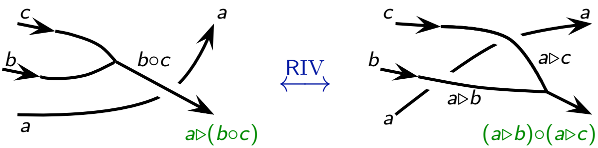

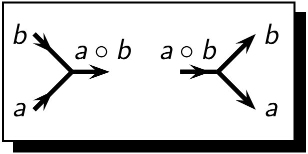

positive branched braid invariants $\overset{\text{colorings}}{\leftsquigarrow} A_n$

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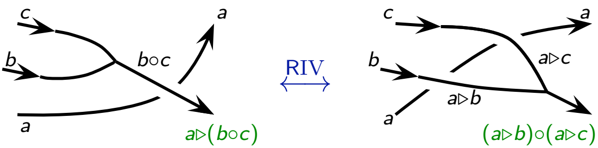

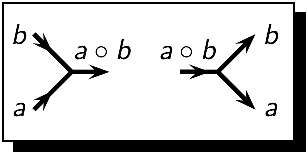

positive branched braid invariants $\overset{\text{colorings}}{\rightsquigarrow}$ A_n

⚠ Does not work for $\mathcal{F}_1$!

Division relations for shelves

	$a \mid_r b$ if $b = c \triangleright a$	$a \mid_l b$ if $b = a \triangleright c$
A_n	is a partial ordering $\rightsquigarrow$ a good basis for 2-cocycles	
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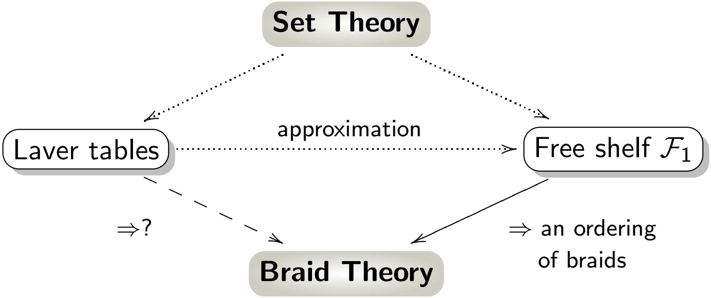
	$a \mid_r b$ if $b = c \triangleright a$	$a \mid_l b$ if $b = a \triangleright c$
A_n	is a partial ordering $\rightsquigarrow$ a good basis for 2-cocycles	induces a trivial relation
$\mathcal{F}_1$	induces a partial ordering $\rightsquigarrow$?	induces a total ordering $\rightsquigarrow$ an ordering of braids

To be continued...

Richard Laver



Patrick Dehornoy