Knotted 3-Valent Graphs, Branched Braids, and Multiplication-Conjugation Relations in a Group

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OCAMI, Osaka

Intelligence of Low-dimensional Topology
Kyoto, May 21-23, 2014

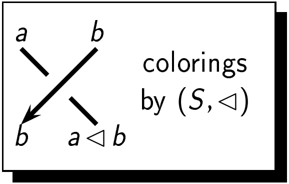
Based on:

✎ Qualgebras and Knotted 3-Valent Graphs, *ArXiv*, 2014

✎ (With S. Kamada) Alexander and Markov Theorems for Graph-Braids, *in progress*

Part 1:
How a Knot Theorist Would Invent Qualgebras

Quandle colorings of knots



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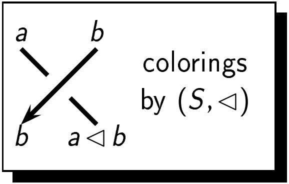

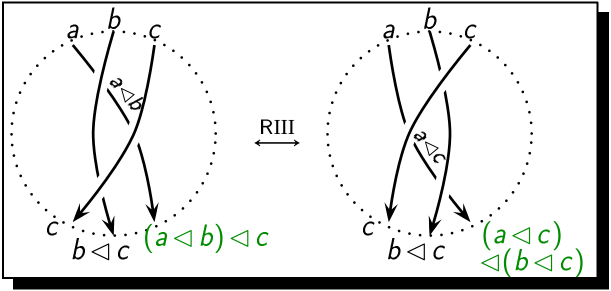

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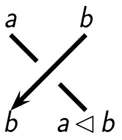
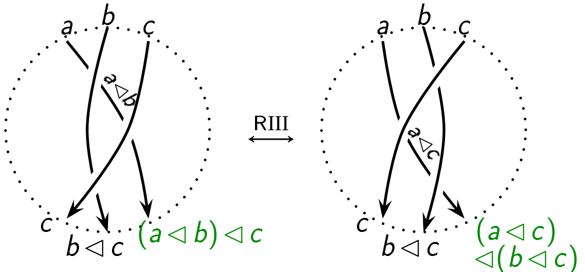


colorings by $(S, \lhd)$



RIII $\leftrightarrow$ $(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$ (SD)

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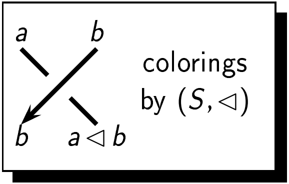

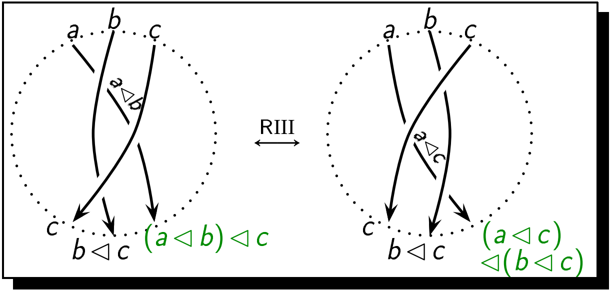

RIII	$\leftrightarrow$	$(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$	(SD)
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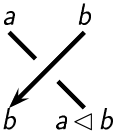

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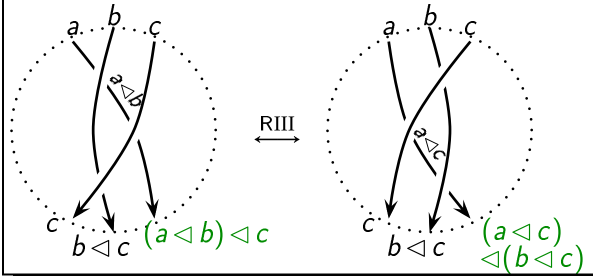

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Example

Group $G \rightsquigarrow$
$Conj(G) = (G, g \lhd h = h^{-1}gh)$.

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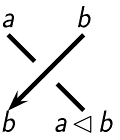

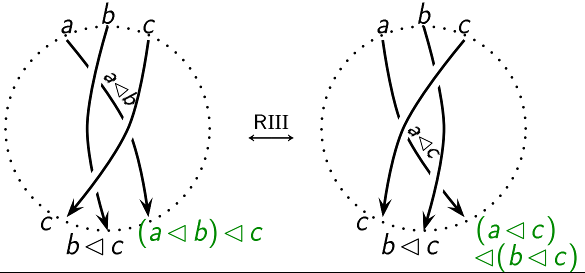

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knot invariants $\overset{\text{colorings}}{\leftsquigarrow}$ quandle

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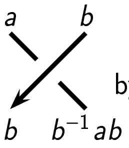
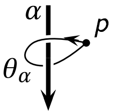


colorings by $Conj(G)$

Wirtinger presentation:



colorings by $Conj(G)$

$\updownarrow$

$Rep(\pi_1(\mathbb{R}^3 \setminus K), G)$

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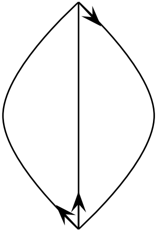
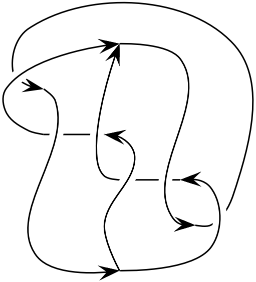
Knotted 3-valent graphs

Standard and Kinoshita-Terasaka Θ-curves:



Θ_{st}



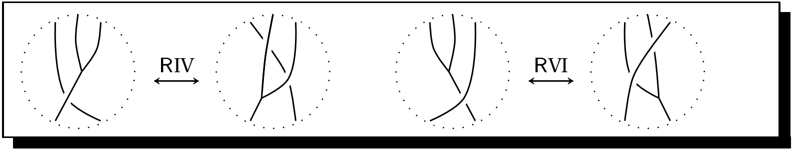
Θ_{KT}

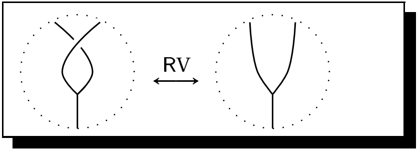
Applications:

- handlebody-knots;
- foams (categorification, 3-manifolds).

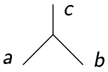
Reidemeister moves for 3-graphs

1989 L.H. Kauffman, S. Yamada, D.N. Yetter:





Extending quandle colorings to 3-graphs?



G is a group;
$a^{\pm 1} b^{\pm 1} c^{\pm 1} = 1$

generalizations:
G-family of quandles (2012 Ishii-Iwakiri-Jang-Oshiro),
multiple conjugation quandle (2013 A. Ishii)

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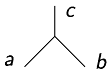
- 

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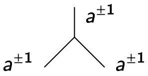
- 

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a *vertex constant version* of $\pi_1(\mathbb{R}^3 \setminus \Gamma)$ (1995 C. Livingston)

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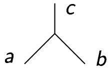
- 

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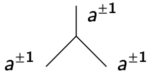
- 

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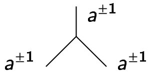
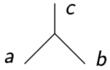
- 

S is a *symmetric* quandle; $\forall x, ((x \lhd a) \lhd a) \lhd a = x$

(2007 Fleming-Mellor)

Extending quandle colorings to 3-graphs?

- 

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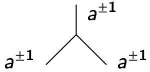
- 

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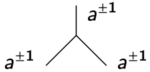
- 

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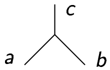
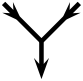
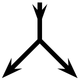
- 

S is a quandle; $\forall x, ((x \lhd^{\pm 1} a) \lhd^{\pm 1} b) \lhd^{\pm 1} c = x$

  (2010 M. Niebrzydowski)

Orientation

Well-oriented 3-graphs: only zip  and unzip  vertices.

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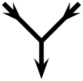
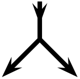
Well-oriented 3-graphs: only zip  and unzip  vertices.

Proposition: Every 3-graph admits a well-orientation.

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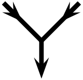
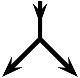
Well-oriented 3-graphs: only zip  and unzip  vertices.

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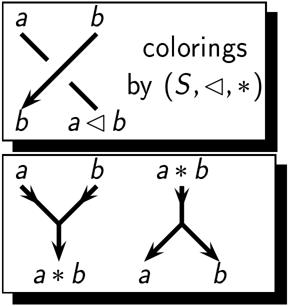
Unoriented 3-graph $\longmapsto$ { its well-orientations }.

Qualgebra colorings for 3-graphs



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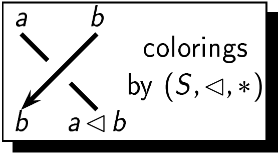

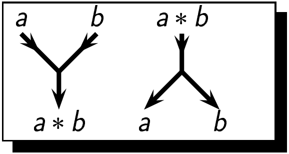

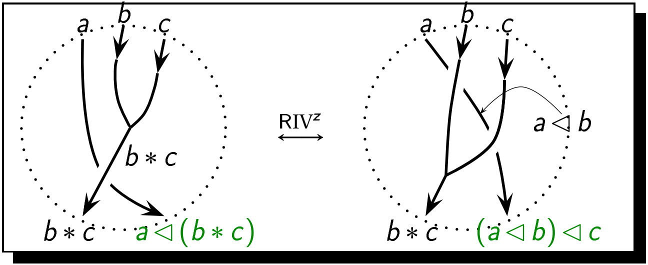

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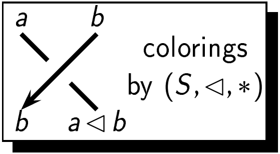

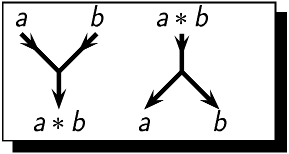

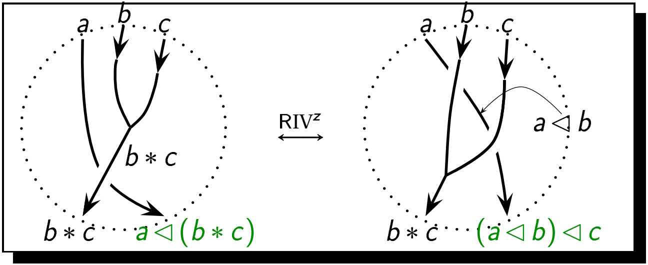

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RV	$\leftrightarrow$	$a * b = b * (a \lhd b)$

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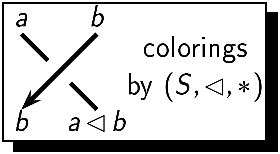

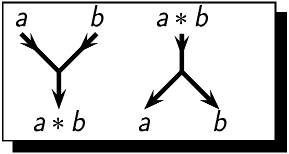

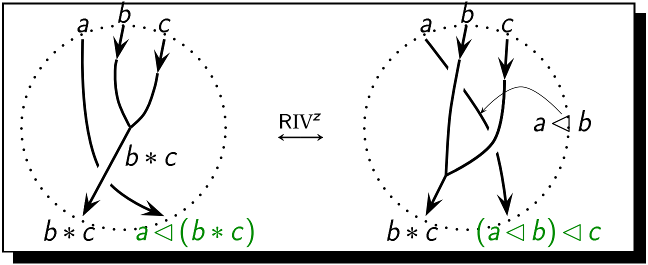

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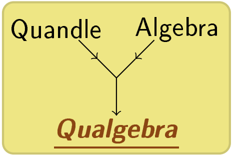

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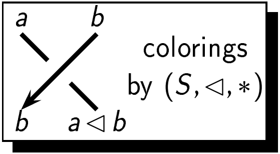

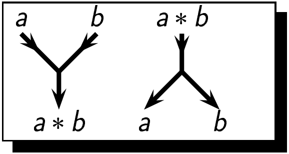

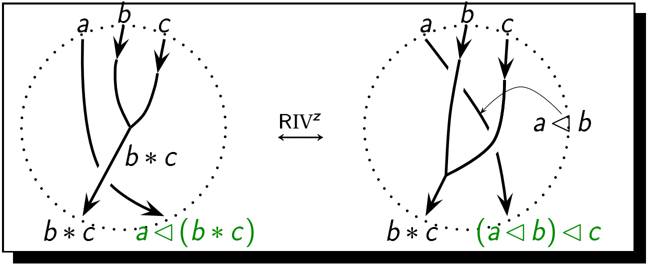

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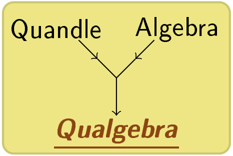
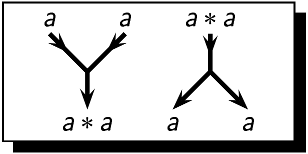


3-graph invariants $\overset{\text{colorings}}{\leftsquigarrow}$ qualgebra

Computation example

Isosceles colorings:



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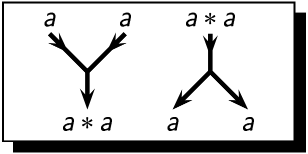
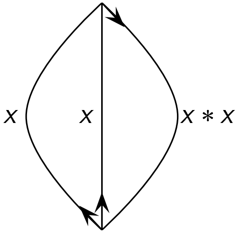
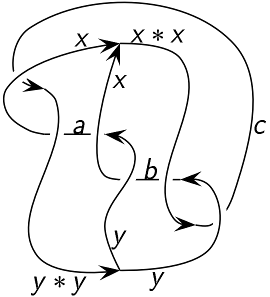


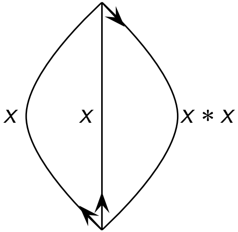


Θ_{st}



Θ_{KT}

Computation example



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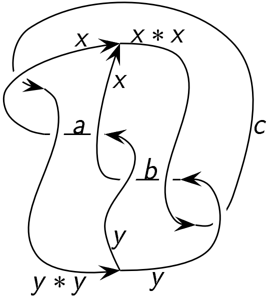

Θ_{KT}

$$(\star)\begin{cases} a &= x \lhd (y * y) = y \lhd x, \\ b &= x \widetilde{\lhd} y = y \widetilde{\lhd} (x * x), \\ c &= (y * y) \lhd x = (x * x) \widetilde{\lhd} y. \end{cases}$$

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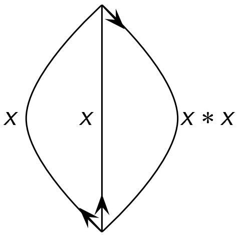

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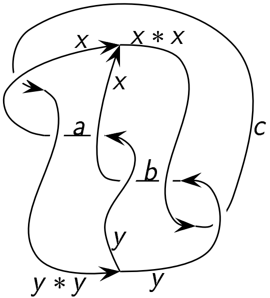

Θ_{KT}

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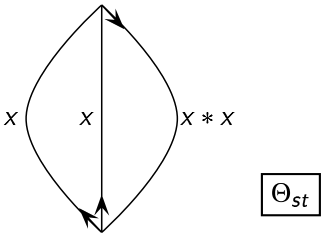
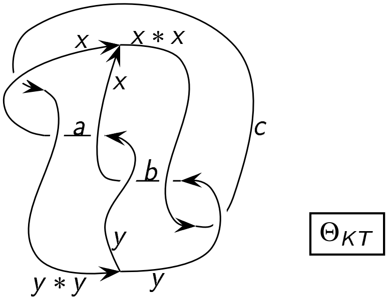
Qualgebra $(S_4, g \lhd h = h^{-1}gh, g * h = gh) \quad \rightsquigarrow \quad (\star) \Leftrightarrow xyx = yxy.$

Computation example





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Qualgebra $(S_4, g \lhd h = h^{-1}gh, g * h = gh) \quad \rightsquigarrow \quad (\star) \Leftrightarrow xyx = yxy$.
Solutions: $x = y$ and $x = (123), y = (432)$ and...
So, $\#\mathscr{C}^{iso}_{S_4}(\Theta_{KT}) > \#S_4 = \#\mathscr{C}^{iso}_{S_4}(\Theta_{st})$.

Part 2:
How an Algebraist Would Invent Qualgebras

Group qualgebras

Example 1

Group $G \rightsquigarrow$ ***group qualgebra*** $QA(G) = (G, g \lhd h = h^{-1}gh, g * h = gh)$.

Group qualgebras

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$$QA(G)\text{-colorings} \xleftrightarrow[\text{presentation}]{\text{Wirtinger}} Rep(\pi_1(\mathbb{R}^3 \setminus \Gamma), G)$$

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abstract level	quandle axioms	specific qualgebra axioms
topology	moves RI-RIII	moves RIV-RVI
groups	conjugation	conjugation-multiplication interaction

Group qualgebras

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abstract level	quandle axioms	specific qualgebra axioms
topology	moves RI-RIII	moves RIV-RVI
groups	conjugation	conjugation-multiplication interaction

Quandle axioms $\Rightarrow$ all properties of conjugation.

⚠ Qualgebra axioms $\not\Rightarrow$ all properties of conjugation/multiplication interaction.

$$(b \lhd a) * (a \lhd b) = ((a \tilde{\lhd} b) \lhd a) * b$$

$$(= a^{-1}bab^{-1}ab)$$

Other qualgebra examples

Example 1

Group $G \rightsquigarrow$ group qualgebra $QA(G) = (G, g \lhd h = h^{-1}gh, g * h = gh)$.

Example 1'

Group G & $X \subset G \rightsquigarrow$ the sub-qualgebra of $QA(G)$ generated by X.

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Example 0

Trivial qualgebra $(S, a \lhd b = a, a * b)$, where $*$ is commutative.

Other qualgebra examples

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Trivial qualgebra $(S, a \lhd b = a, a * b)$, where $*$ is commutative.

$\rightsquigarrow$ abstract graph invariants

"Qualgebrability"

Existence

Dihedral quandle $(\mathbb{Z}/n\mathbb{Z}, a \lhd b = 2b - a)$ is not "qualgebrizable":

$$(a \lhd b) \lhd c = 2c - 2b + a$$
$$a \lhd (b * c) = 2(b * c) - a$$

"Qualgebrability"

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Uniqueness

$QA(S_3)$ is the unique "qualgebraization" of $Conj(S_3)$.

Small examples

Example 4

Non-trivial qualgebra structures on $Q = \{p, q, r, s\}$?

Description: put $\overline{p} = q, \overline{q} = p, \overline{r} = r, \overline{s} = s$;

$x \lhd r = \overline{x}, \qquad x \lhd y = x$ for other y;

$*$ is commutative,

$r * x = r$ for $x \neq r, \qquad r * r = s * s = s,$

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- Not cancellative ⇒ do not come from groups.

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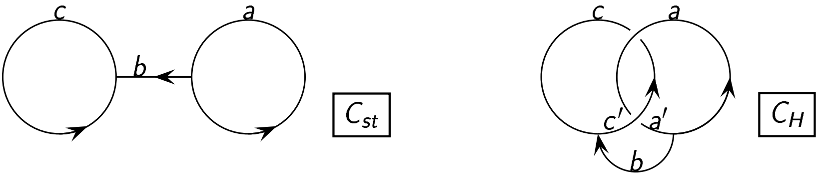
- Not cancellative ⇒ do not come from groups.
- Two are associative, three have neutral elements, none are unital associative.

Computation example

Standard and Hopf cuff graphs:



$$\#\mathscr{C}_Q(C_{st}) = \#\{(a,b,c) \in Q \,|\, b * a = a,\ b * c = c\} = 18,$$
$$\#\mathscr{C}_Q(C_H) = \#\{(a,b,c) \in Q \,|\, b * a = a \lhd c,\ b * c = c \lhd a\} = 14.$$

Part 3:
Variations on
Qualgebra Ideas

Symmetric qualgebras and orientation independence

Problem: Qualgebra invariants depend on orientations.

Symmetric qualgebras and orientation independence

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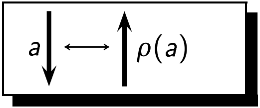

Solution: a ***good involution*** $\rho : S \to S$.

Symmetric qualgebras and orientation independence

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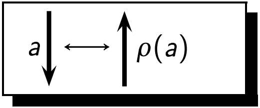

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$$\rho(\rho(a)) = a$$
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$$a \lhd \rho(b) = a \widetilde{\lhd} b$$
$$(a * b) * \rho(b) = \rho(b) * (b * a) = a$$

} ***Symmetric quandle*** (1996 S. Kamada) } ***Symmetric qualgebra***

Symmetric qualgebras and orientation independence

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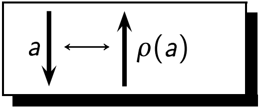

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$$\begin{aligned}
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&a \lhd \rho(b) = a \widetilde{\lhd} b\\
&(a * b) * \rho(b) = \rho(b) * (b * a) = a
\end{aligned}$$

The first three conditions: *Symmetric quandle* (1996 S. Kamada)

All four conditions: *Symmetric qualgebra*

Example

Group $G \rightsquigarrow QA^*(G) = (G, g \lhd h = h^{-1}gh, g * h = h, \rho(h) = h^{-1})$.

Symmetric qualgebras and orientation independence

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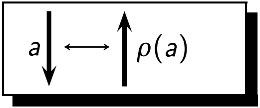

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The first three conditions give a ***Symmetric quandle*** (1996 S. Kamada); all four give a ***Symmetric qualgebra***.

Example

Group $G \rightsquigarrow QA^*(G) = (G, g \lhd h = h^{-1}gh, g * h = h, \rho(h) = h^{-1})$.

abstract level	specific symmetric qualgebra axioms
topology	unoriented 3-graphs
groups	conjugation- and multiplication-inversion interaction

Symmetric qualgebras: examples

$\rho(\rho(a)) = a$
$\rho(a) \lhd b = \rho(a \lhd b)$
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Properties

- $\forall\, b$, maps $a \mapsto a * b$ and $a \mapsto b * a$ are bijections. $\rightsquigarrow$ *pseudo-sudoku*
- Symmetric qualgebras with associative $*$ $\longleftrightarrow$ groups.

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- $\forall\, b$, maps $a \mapsto a * b$ and $a \mapsto b * a$ are bijections. $\rightsquigarrow$ *pseudo-sudoku*
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Example 3

$S = \{x, y, z\}$, $a \lhd b = a$, $*$ is commutative.

$*$	x	y	z
x	x	y	z
y	y	z	x
z	z	x	y
ρ	x	z	y

$\mathbb{Z}/3\mathbb{Z}$

$*$	x	y	z
x	x	z	y
y	z	y	x
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not groups

Symmetric qualgebras: examples

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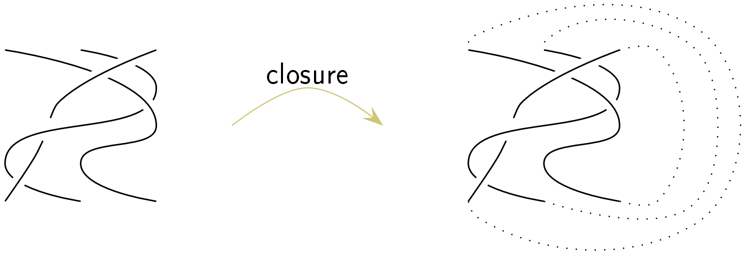
(the last two:) not groups

Example 4

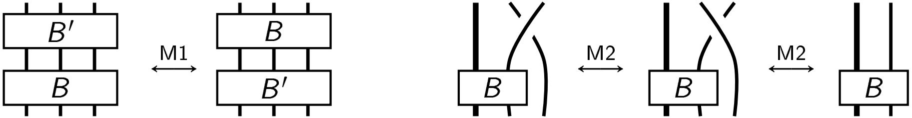
- Non-trivial qualgebras of size 4 are not symmetric.
- Trivial qualgebras: $QA^*(\mathbb{Z}/4\mathbb{Z})$, $QA^*(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$, and two non-associative ones.

Alexander-Markov theorem



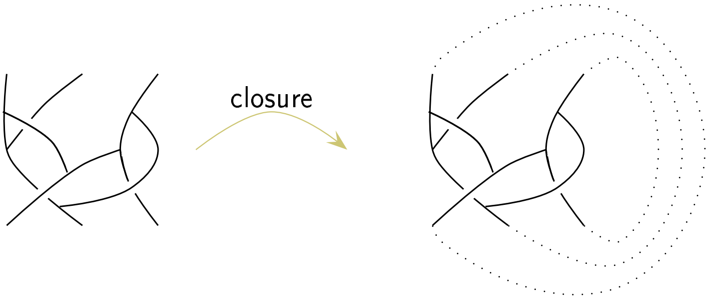
Theorem (Alexander, 1923; Markov, 1935)

- Surjectivity.
- Kernel: moves M1 and M2.



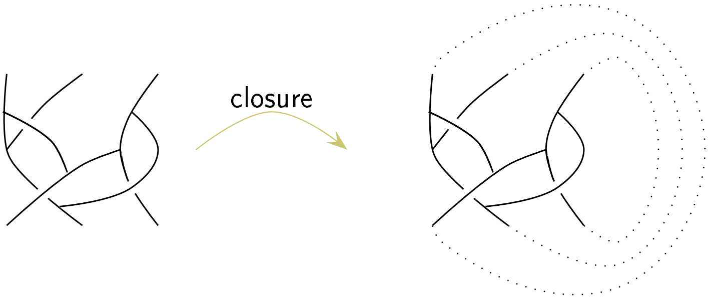
Branched Alexander-Markov theorem

Branched braids



Branched Alexander-Markov theorem

Branched braids



Theorem (S. Kamada - L., 2014)

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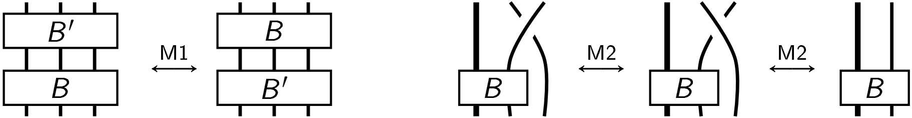

Branched Alexander-Markov theorem

$$\text{branched braids} \xrightarrow{\text{closure}} \text{3-graphs}$$

Theorem (S. Kamada - L., 2014)

- Surjectivity.
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Generalizations

- Graph-braids (vertices of arbitrary valence).
- Virtual and welded versions.

Branched Alexander-Markov theorem

branched braids $\stackrel{\text{closure}}{\longrightarrow}$ 3-graphs

Theorem (S. Kamada - L., 2014)

- Surjectivity.
- Kernel: moves M1 and M2.

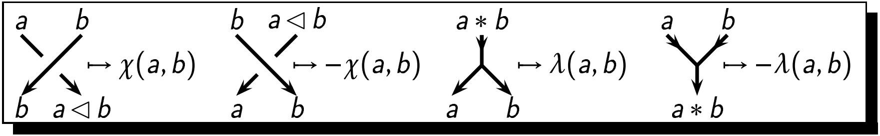
Generalizations

- Graph-braids (vertices of arbitrary valence).
- Virtual and welded versions.

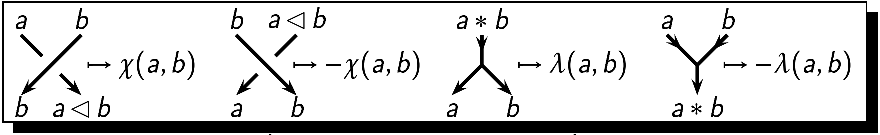
branched braid invariants $\stackrel{\text{colorings}}{\leftsquigarrow}$ weak qualgebra

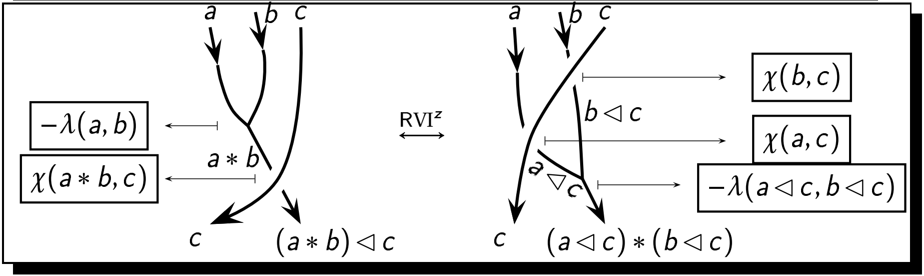
(omit $a \lhd a = a$)

Qualgebra cocycle invariants for 3-graphs



Qualgebra cocycle invariants for 3-graphs





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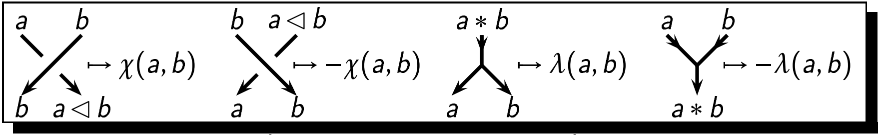
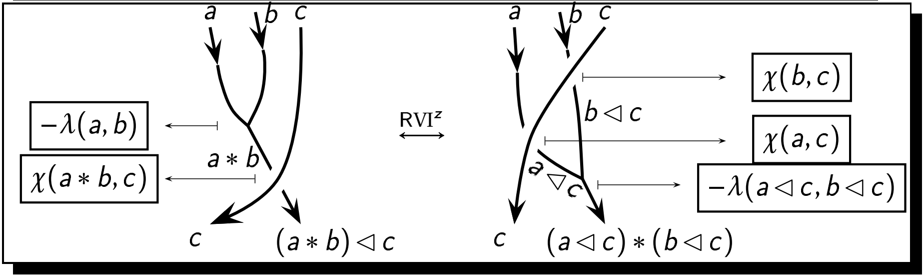


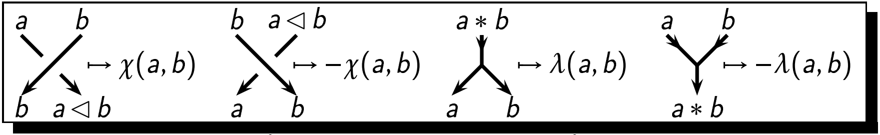


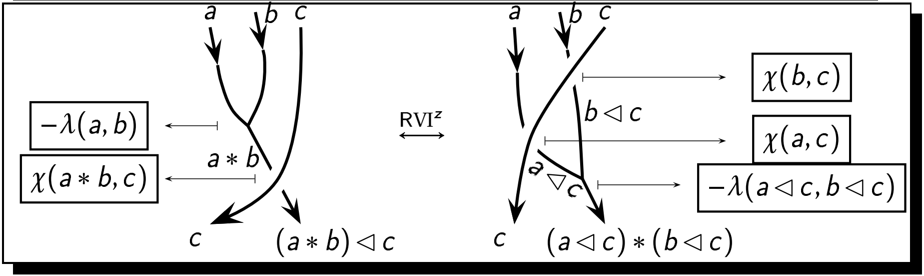
RIV $\leftrightarrow$ $\chi(a, b * c) = \chi(a, b) + \chi(a \lhd b, c)$

RVI $\leftrightarrow$ $\chi(a * b, c) + \lambda(a \lhd c, b \lhd c) = \chi(a, c) + \chi(b, c) + \lambda(a, b)$

RV $\leftrightarrow$ $\chi(a, b) + \lambda(a, b) = \lambda(b, a \lhd b)$

Qualgebra cocycle invariants for 3-graphs





RIV $\leftrightarrow$	$\chi(a, b*c) = \chi(a,b) + \chi(a \lhd b, c)$
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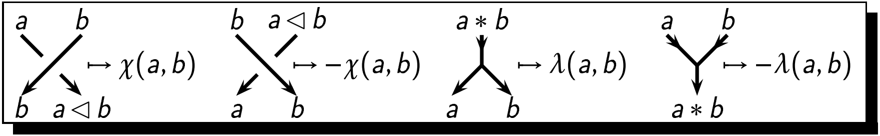
RV $\leftrightarrow$	$\chi(a,b) + \lambda(a,b) = \lambda(b, a \lhd b)$

} ***Qualgebra*** **2-*cocycle***

RI-RIII are automatic.

Qualgebra cocycle invariants for 3-graphs



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} ***Qualgebra* 2-*cocycle***

RI-RIII are automatic.

3-graph invariants $\overset{\text{colorings}}{\underset{\text{weights}}{\leftsquigarrow}}$ qualgebra & 2- or 3-cocycle

Qualgebra cocycles: example

Example 4

$Q = \{p, q, r, s\}$

$x \lhd r = \overline{x}, \qquad x \lhd y = x$ for other y;

$*$ is commutative,

$r * x = r$ for $x \neq r, \qquad r * r = s * s = s,$

$p * p = \overline{q * q} \in \{p, q, s\}, \qquad p * q = s,$

$p * s = \overline{q * s} \in \{p, q, s\}.$

- $Z^2(Q) \cong \mathbb{Z}^8$
- $B^2(Q) \cong \mathbb{Z}^4$
- $H^2(Q) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^4$

Qualgebra cocycle invariants: diverse questions

Qualgebra* 2-*coboundaries:

$$\phi : S \to \mathbb{Z} \rightsquigarrow \quad \begin{aligned} \chi(a,b) &= \phi(a \lhd b) - \phi(a), \\ \lambda(a,b) &= \phi(a) + \phi(b) - \phi(a * b) \end{aligned} \quad \rightsquigarrow \text{trivial invariants}$$

Qualgebra cocycle invariants: diverse questions

Qualgebra 2-coboundaries:

$$\phi : S \to \mathbb{Z} \rightsquigarrow \qquad \chi(a,b) = \phi(a \lhd b) - \phi(a), \qquad \rightsquigarrow \text{trivial}$$
$$\lambda(a,b) = \phi(a) + \phi(b) - \phi(a * b) \qquad \text{invariants}$$

Question: Define and study qualgebra homology in higher degrees?

Qualgebra cocycle invariants: diverse questions

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$$\phi : S \to \mathbb{Z} \rightsquigarrow \quad \begin{aligned} \chi(a,b) &= \phi(a \lhd b) - \phi(a), \\ \lambda(a,b) &= \phi(a) + \phi(b) - \phi(a * b) \end{aligned} \quad \rightsquigarrow \text{trivial invariants}$$

Question: Define and study qualgebra homology in higher degrees?
Solution: for ***rigid qualgebras*** ($\longleftrightarrow$ rigid 3-graphs).

Qualgebra cocycle invariants: diverse questions

Qualgebra 2-coboundaries:

$\phi : S \to \mathbb{Z} \rightsquigarrow$ $\chi(a,b) = \phi(a \lhd b) - \phi(a),$ $\lambda(a,b) = \phi(a) + \phi(b) - \phi(a * b)$ $\rightsquigarrow$ trivial invariants

Question: Define and study qualgebra homology in higher degrees?
Solution: for ***rigid qualgebras*** ($\longleftrightarrow$ rigid 3-graphs).

Enhancements

- Region coloring and shadow cocycle invariants.

Qualgebra cocycle invariants: diverse questions

Qualgebra 2-coboundaries:

$\phi : S \to \mathbb{Z} \rightsquigarrow$ $\chi(a,b) = \phi(a \triangleleft b) - \phi(a),$ $\rightsquigarrow$ trivial

$\lambda(a,b) = \phi(a) + \phi(b) - \phi(a * b)$ invariants

Question: Define and study qualgebra homology in higher degrees?
Solution: for ***rigid qualgebras*** ($\longleftrightarrow$ rigid 3-graphs).

Enhancements

- Region coloring and shadow cocycle invariants.
- Distinguish zip- and unzip-vertices: