Self-, Multi- and *G*-Distributivity with a Braided Flavor

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OCAMI, Osaka

Special Session
Algebraic Structures Motivated by Knot Theory
JMM, January 16, 2014

Overview

algebraic structures $\xleftrightarrow[\text{motivated by}]{\text{motivated by}}$ knot theory

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algebraic structures $\overset{\text{motivated by}}{\underset{\text{motivated by}}{\longleftrightarrow}}$ knot theory

Multi-braidings (= families of YB operators):

Braids and knots:

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Braids and knots:

- shelves

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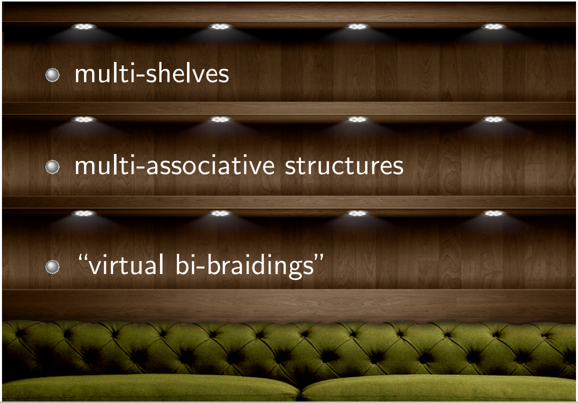

- multi-shelves
- multi-associative structures
- "virtual bi-braidings"

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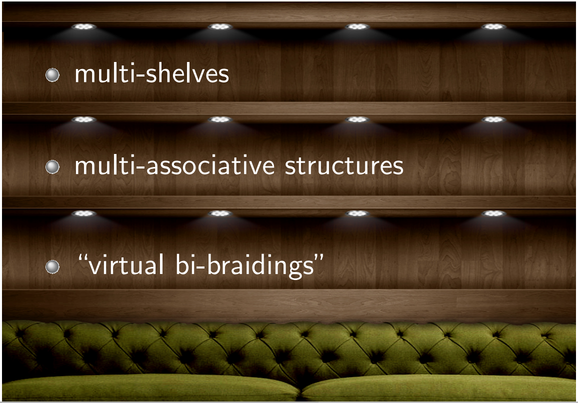
Multi-braidings (= families of YB operators):



- multi-shelves
- multi-associative structures
- “virtual bi-braidings”

Braids and knots:

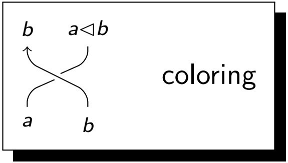
- (bi-)colorings
- counting and cocycle invariants

Shelves

Algebraic structures

$(S, \lhd)$

Knot theory



Shelves

Algebraic structures

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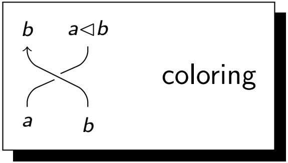
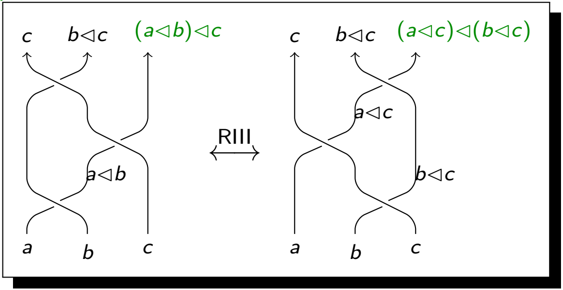
$(S, \triangleleft)$ s.t.



$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$

(Self-Distributivity)

shelf



Multi-shelves

Algebraic structures

$(S, \{\lhd_i\}_{i \in \mathbf{I}})$ s.t.

$$(a \lhd_j b) \lhd_k c = (a \lhd_k c) \lhd_j (b \lhd_k c)$$

(Multi-Distributivity)

$$\underbrace{\qquad\qquad\qquad\qquad}$$

multi-shelf

(P.Dehornoy, D.Larue; J.Przytycki, A.Sikora, K.Putyra)

Multi-shelves

Algebraic structures

Knot theory

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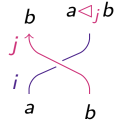
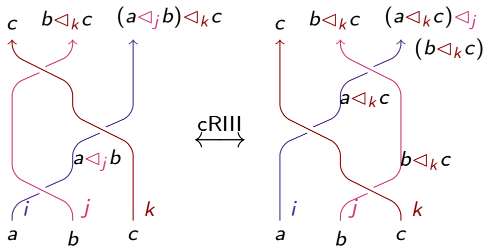


$$(a \lhd_j b) \lhd_k c = (a \lhd_k c) \lhd_j (b \lhd_k c)$$

(Multi-Distributivity)



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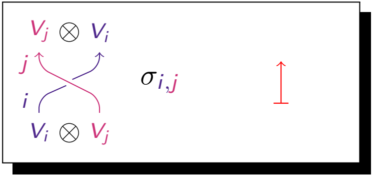
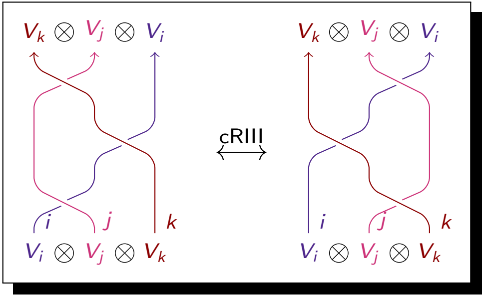
Braided systems: definition

Algebraic structures

Knot theory





Braided systems: definition

Algebraic structures

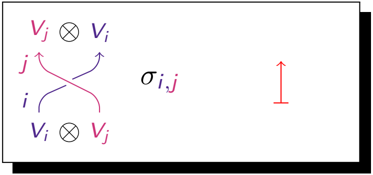
$(\{V_i\}_{i \in \mathbf{I}}, \{\sigma_{i,j}\}_{i,j \in \mathbf{I}})$ s.t.

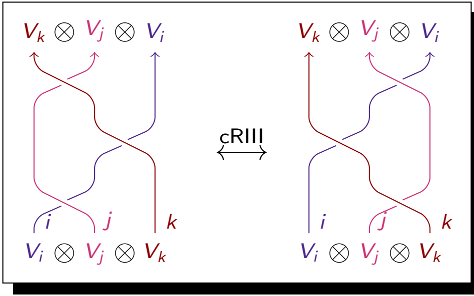
$$(\sigma_{j,k} \otimes \mathrm{Id}_i) \circ (\mathrm{Id}_j \otimes \sigma_{i,k}) \circ (\sigma_{i,j} \otimes \mathrm{Id}_k) = (\mathrm{Id}_k \otimes \sigma_{i,j}) \circ (\sigma_{i,k} \otimes \mathrm{Id}_j) \circ (\mathrm{Id}_i \otimes \sigma_{j,k})$$

(Colored Yang-Baxter Equation)

braided system

Knot theory





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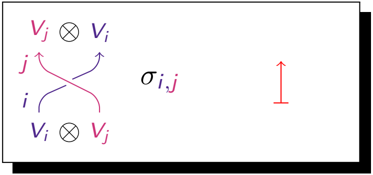
$$(\{V_i\}_{i\in \mathbf{I}}, \{\sigma_{i,j}\}_{\substack{i,j\in\mathbf{I}\\ i\leqslant j}})\ \text{s.t.}$$

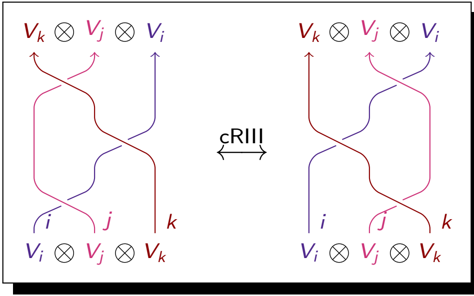
$$(\sigma_{j,k}\otimes \mathrm{Id}_i)\circ(\mathrm{Id}_j\otimes\sigma_{i,k})\circ(\sigma_{i,j}\otimes \mathrm{Id}_k) =$$
$$(\mathrm{Id}_k\otimes\sigma_{i,j})\circ(\sigma_{i,k}\otimes \mathrm{Id}_j)\circ(\mathrm{Id}_i\otimes\sigma_{j,k})$$
$$\forall\ i\leqslant j\leqslant k$$

(Colored Yang-Baxter Equation)

(ordered) braided system

Knot theory





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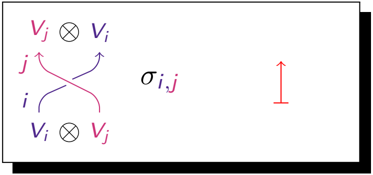
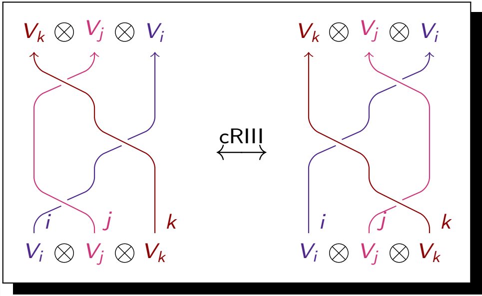
Example (Multi-shelves)

$V_i = S$, $\sigma_{i,j} : (a,b)\mapsto(b, a\lhd_j b)$.

Knot theory





Braided systems: representation theory

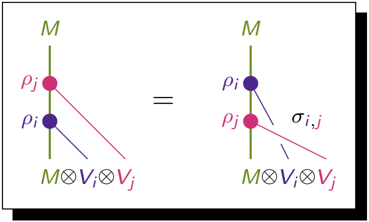
Algebraic structures

$(M, \{\rho_i : M \otimes V_i \to M\}_{i \in \mathbf{I}})$ s.t.

$$\rho_j \circ (\rho_i \otimes \mathrm{Id}_j) = \rho_i \circ (\rho_j \otimes \mathrm{Id}_i) \circ (\mathrm{Id}_M \otimes \sigma_{i,j})$$
$$\forall\, i \leqslant j$$

multi-braided module

Knot theory



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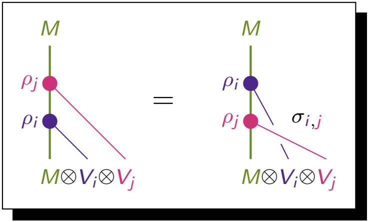

Example (Multi-shelves)

$$(m \blacktriangleleft_i a) \blacktriangleleft_j b = (m \blacktriangleleft_j b) \blacktriangleleft_i (a \triangleleft_j b)$$
(multi-S-set).

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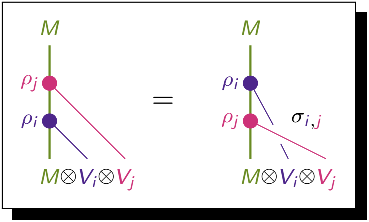

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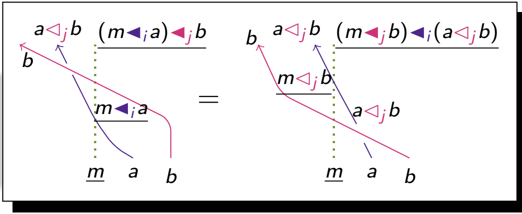


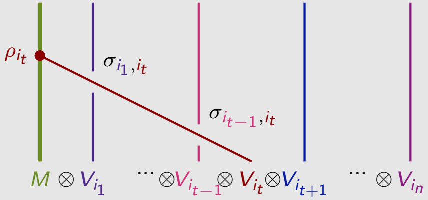
Braided systems: homology theory

Theorem (L., 2012)

Take an ordered braided system of abelian groups $(\{V_1, V_2, \ldots\}, \{\sigma_{i,j}\}_{i \leqslant j})$ *and a multi-braided module* $(M, \{\rho_i : M \otimes V_i \to M\}_{i=1,2,\ldots})$ *over it. A differential on* $M \otimes T(V_1) \otimes T(V_2) \otimes \ldots$ *can then be defined as follows:*

$$d_n := \sum_{t=1}^{n} (-1)^{t-1}\, d_{n;t},$$

$d_{n;t} =$



$i_1 \leqslant i_2 \leqslant \ldots \leqslant i_n$

Braided systems: homology theory

Proof

Pre-simplicial axiom

$$d_{n-1;j} \circ d_{n;k} = d_{n-1;k-1} \circ d_{n;j} \qquad 1 \leqslant j < k \leqslant n.$$

$+$ a careful sign juggling.

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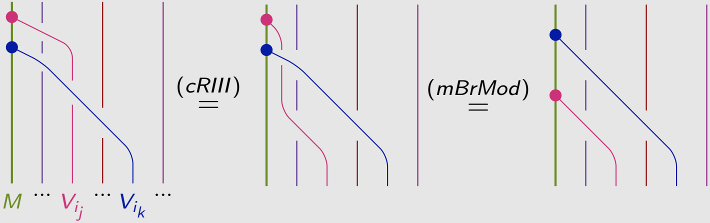

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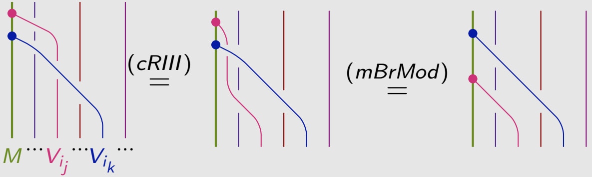
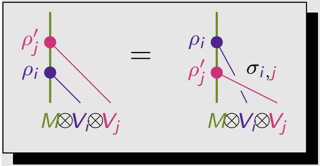


Remarks

➻ "Compatible" multi-braided modules $\rightsquigarrow$ compatible differentials $\rightsquigarrow$ linear combinations are differentials.



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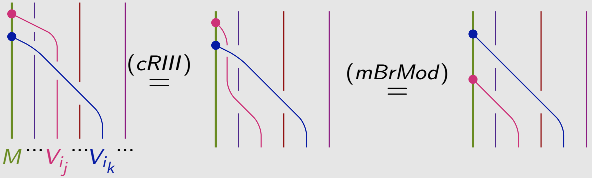

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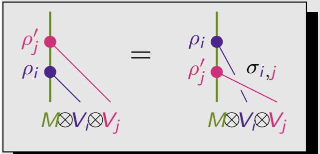

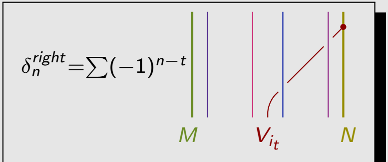

Braided systems: homology theory

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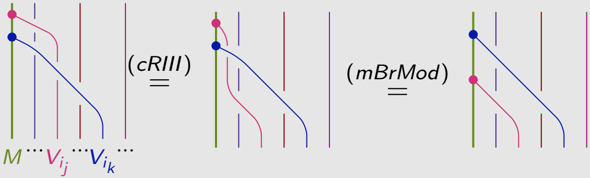

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Braided systems: homology theory

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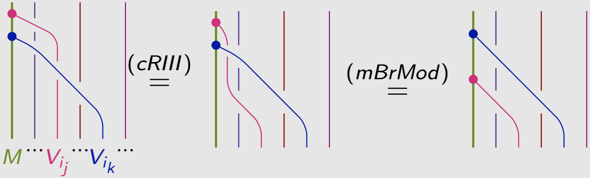

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➵ Functoriality.

➵ Homology morphisms.

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Example (Shelves)

Linearization: $V = \mathbb{Z}S$; a rack-set M (or $M = \mathbb{Z}$, $\rho : a \mapsto 1$).

➜ $\delta^{left} \rightsquigarrow$ *shelf / 1-term distributive homology* (2011, Przytycki-Sikora).

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➜ $\Delta(a) = a \otimes a \rightsquigarrow$ degeneracies $\rightsquigarrow$ *quandle homology* (2003, Carter-Jelsovsky-Kamada-Langford-Saito).

Examples: Multi-shelves

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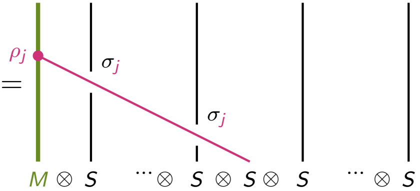
Take a multi-S-set $(M, \{\blacktriangleleft_i : M \times S \to M\}_{i \in \mathbf{I}})$.
Then the maps

$$d^j_n := \sum_{t=1}^{n} (-1)^{t-1} \, d^j_{n;t}, \qquad d^j_{n;t} =$$



define a multi-differential on $\mathbb{Z}M \times S^n$.

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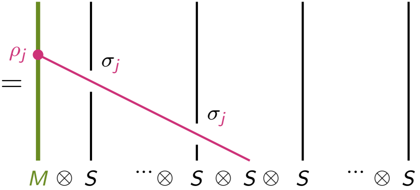

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$\rightsquigarrow$ linear combinations are differentials

$\rightsquigarrow$ *multi-term distributive homology* (2011, Przytycki-Sikora)

Examples: (multi-)associativity

Unitary associative algebras

$(V, \cdot, \mathbf{1}) \rightsquigarrow$ a braided system $(V, \sigma_{Ass} : v \otimes w \mapsto \mathbf{1} \otimes v \cdot w)$.

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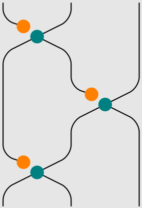

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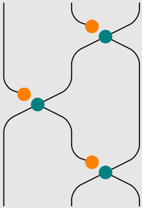

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YBE for $\sigma_{Ass} \iff$ associativity for $\cdot$

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$$\begin{aligned} \text{YBE for } \sigma_{Ass} &\iff \text{associativity for } \cdot \\ \text{braided module over } (V, \sigma_{Ass}) &\iff \text{algebra module over } (V, \cdot, \mathbf{1}) \\ \delta^{left} &\longleftrightarrow \text{bar complex} \\ \delta^{left} - \delta^{right} \mod \text{degeneracies} &\longleftrightarrow \text{Hochschild complex} \end{aligned}$$

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Bialgebras

$(H, \mu, \mathbf{1}, \Delta, \varepsilon)$

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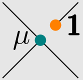
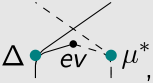
Bialgebras

$(H, \mu, \mathbf{1}, \Delta, \varepsilon) \rightsquigarrow$ an ordered braided system: $V_1 = H, V_2 = H^*$,

$\sigma_{1,1} =$ 

, $\sigma_{1,2} =$ 

, $\sigma_{2,2} =$ 

.

Examples: (multi-)associativity

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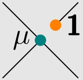
$\sigma_{1,1} =$ 

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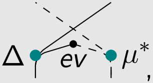
$\sigma_{1,2} =$ 

,

$\sigma_{2,2} =$ 

.

YBE on $H \otimes H \otimes H^* \iff$ the bialgebra compatibility condition

invertibility of $\sigma_{1,2} \iff H$ is a Hopf algebra

multi-braided module $\iff$ right-right Hopf module over H

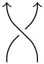
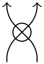
$\delta^{left} - \delta^{right} \longleftrightarrow$ Gerstenhaber-Schack complex

Examples: virtual bi-braidings

V; $\sigma_r, \sigma_v : V^{\otimes 2} \to V^{\otimes 2}$ with $\sigma_v^2 = \mathrm{Id}$.

Proposition:
$(V_1 = V_2 = V,$
$\sigma_{1,1} = \sigma_r, \sigma_{1,2} = \sigma_{2,2} = \sigma_v)$
is an ordered braided system

$\Longleftrightarrow$

 $\mapsto \sigma_r$,  $\mapsto \sigma_v$
defines a representation of positive virtual braid groups.

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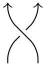
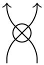
 $\mapsto \sigma_r$,  $\mapsto \sigma_v$

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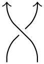
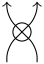
 $\mapsto \sigma_r$,  $\mapsto \sigma_v$

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Theorem $\rightsquigarrow$ a homology theory for virtual bi-braidings.

Digression: cocycle invariants

$$\phi_{i,j} : S_i \times S_j \to \mathbb{Z},\ 1 \leqslant i \leqslant j$$

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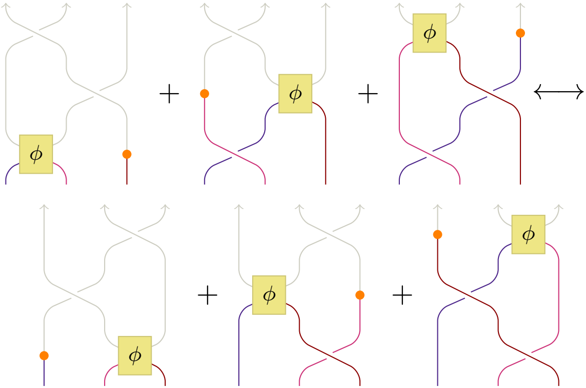

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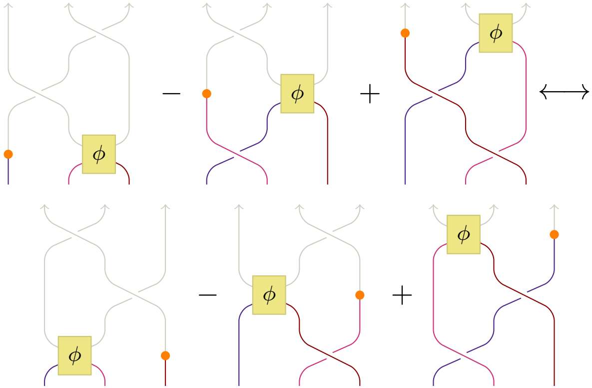

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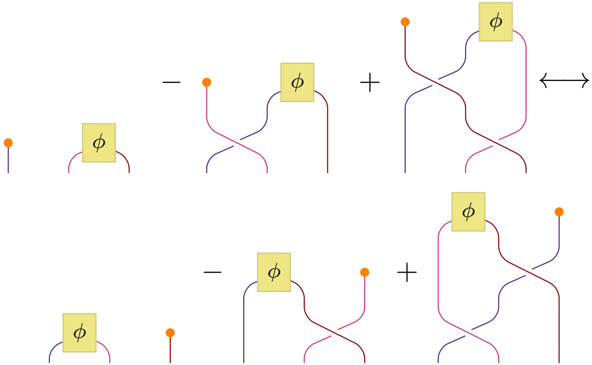

invariance under cRIII $\iff$ ϕ is a 2-cocycle for $\delta^{left} - \delta^{right}$

Examples: virtual bi-braidings

$V = \mathbb{Z}S$; $\sigma_r, \sigma_v : S^{\times 2} \to S^{\times 2}$ s.t. $\sigma_v^2 = \mathrm{Id}$ and
$(S_1 = S_2 = S, \sigma_{1,1} = \sigma_r, \sigma_{1,2} = \sigma_{2,2} = \sigma_v)$ is an ordered braided system;
$\rho_1 = \rho_2 : V \to \mathbb{Z}, a \mapsto 1$, $\delta = \delta^{left} - \delta^{right}$;

2-cocycles: 3 maps $\phi_{i,j} : S \times S \to G$, $1 \leqslant i \leqslant j \leqslant 2$
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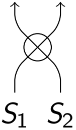

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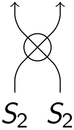

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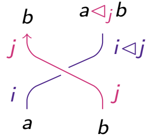
$\rightsquigarrow$ *cocycle invariants for positive virtual braids*
(a conceptual setting for virtual Yang-Baxter cocycle invariants of Ceniceros-Nelson, 2009)

G-distributivity

Knot theory



bi-colorings

G-distributivity

Algebraic structures

a group G & $(S, \{\triangleleft_i\}_{i \in G})$ s.t.

$$(a \triangleleft_j b) \triangleleft_k c = (a \triangleleft_k c) \triangleleft_{k^{-1}jk} (b \triangleleft_k c)$$

(G-Distributivity)

G-family of shelves

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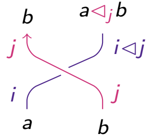

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(2012, Ishii-Iwakiri-Jang-Oshiro)

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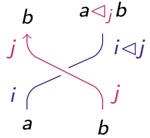

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