Categorical aspects of virtuality

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OCAMI, Osaka City University

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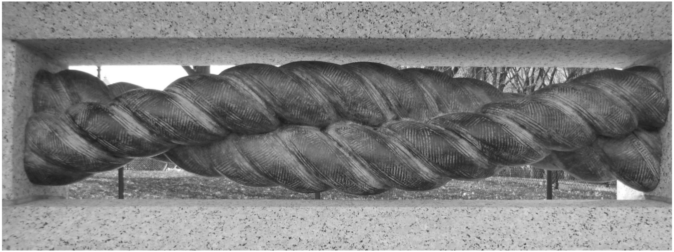


Plan

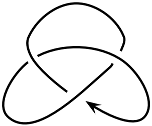
1. Different viewpoints on usual and virtual braids
2. A categorical interpretation of virtual braids
3. Categorical self-distributivity
4. A categorical interpretation of branched braids

Outline

braids	usual	virtual
Topology	known	known
Algebra	known	known
Representation Theory	known	known & **new**
Category Theory	known	**new**

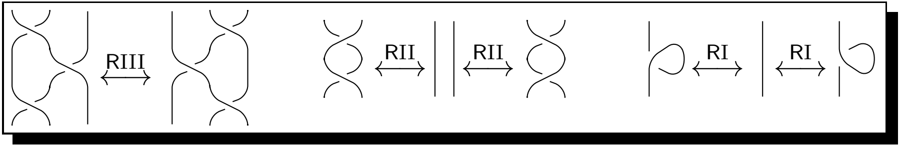
Knots, diagrams and Gauss codes

Knots



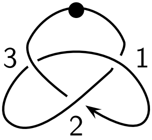
K. Reidemeister, '26:

Knots $\cong$ Diagrams / RI-RIII



Knots, diagrams and Gauss codes

Knots $\xrightarrow{\text{encoding}}$ **Gauss code**



$\longmapsto$ $1U^+2O^+3U^+1O^+2U^+3O^+$

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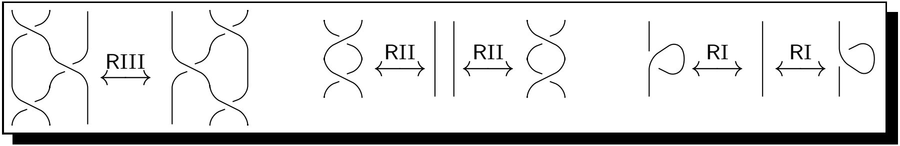
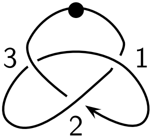


Knots, diagrams and Gauss codes

Knots — encoding → / ← decoding? — **Gauss code**



$$\longmapsto \quad 1U^+2O^+3U^+1O^+2U^+3O^+$$

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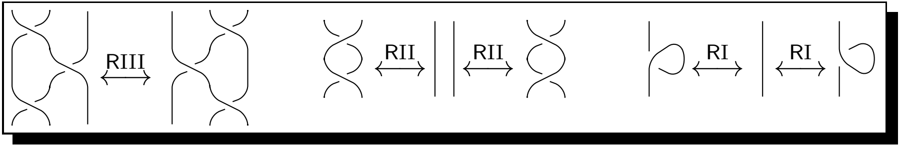
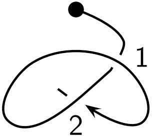
$$\text{Knots} \cong \text{Diagrams} \,/\, \text{RI-RIII}$$



Virtual knots

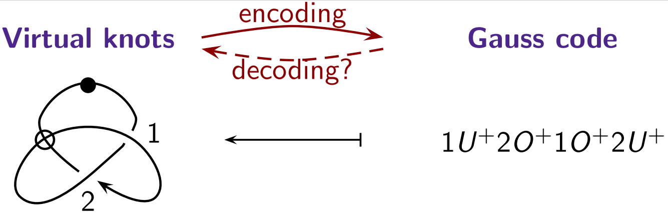
Knots —encoding→ / ←decoding?— **Gauss code**



$\longleftarrow\!\!\dashv$ $1U^+2O^+1O^+2U^+$

Problem: decoding is not surjective.

Virtual knots



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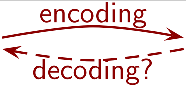
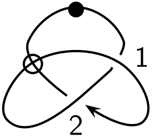
L.H. Kauffman, ’96:

Virtual knots

Virtual knots



Gauss code



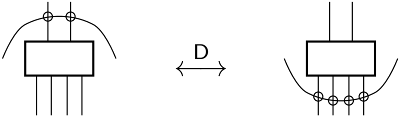
$\longleftarrow\!\!\!\shortmid$

$1U^{+}2O^{+}1O^{+}2U^{+}$

Problem: decoding is not surjective.

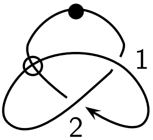
L.H. Kauffman, '96: Virtual knots $\stackrel{Def.}{=}$ Diagrams / RI-RIII, D

Detour move:



Virtual knots

Virtual knots $\underset{\text{decoding}}{\overset{\text{encoding}}{\rightleftarrows}}$ **Gauss code**



$\longleftarrow\!\!\!\shortmid$ $1U^+2O^+1O^+2U^+$

Problem: decoding is not surjective.

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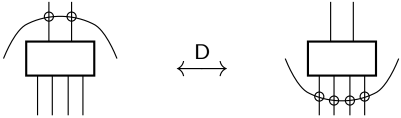
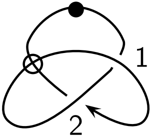


Virtual knots

Virtual knots $\xrightarrow{\text{encoding}}$ **Gauss code**
Virtual knots $\xleftarrow{\text{decoding}}$ **Gauss code**

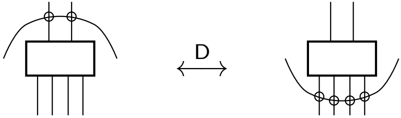


$\longleftarrow\!\shortmid$ $\quad 1U^{+}2O^{+}1O^{+}2U^{+}$

Problem: decoding is not surjective.

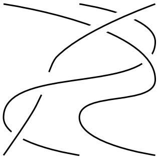
L.H. Kauffman, '96: $\text{Virtual knots} \overset{Def.}{=} \text{Diagrams} \big/ \text{RI-RIII, D}$

Detour move:



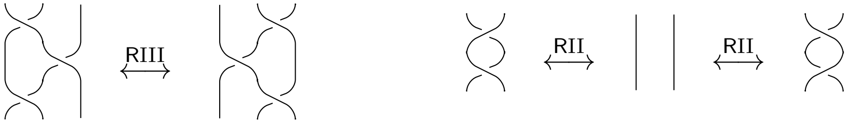
$\text{Virtual knots} \cong \text{Gauss codes} \big/ \text{RI-RIII}$

Braids in topology

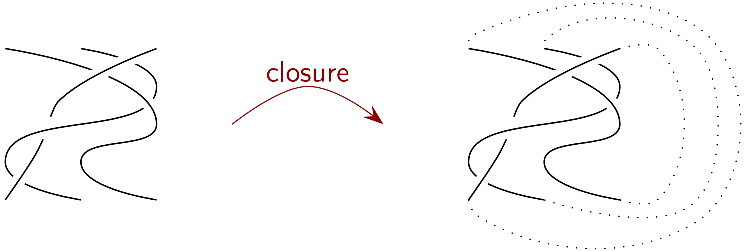


E. Artin, '25:

Braids $\cong$ Diagrams / RII-RIII

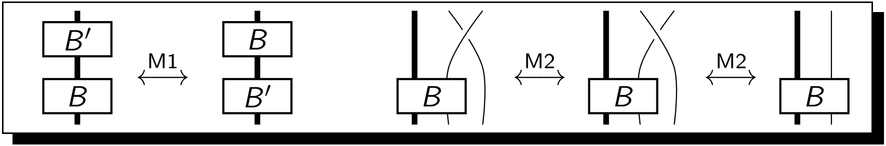


Braids and knots



Theorem (J.W. Alexander, '23; A. Markov, '35)

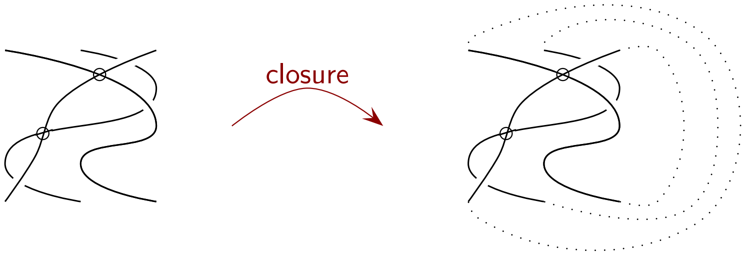
- Surjectivity.
- Kernel: moves M1 and M2.



Virtual knots as closures?

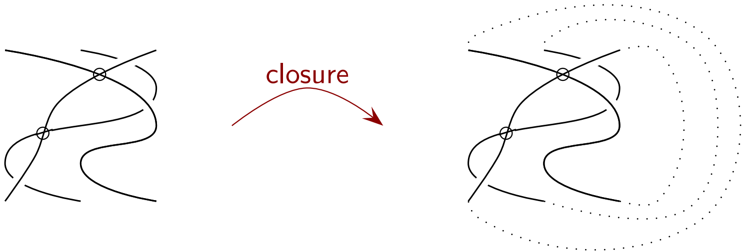
Virtual braids? —closure→ **Virtual knots**

Virtual knots as closures?



V.V. Vershinin, '01:

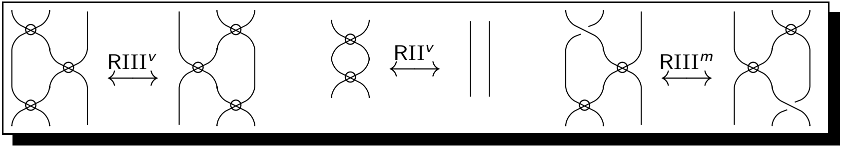
Virtual knots as closures?



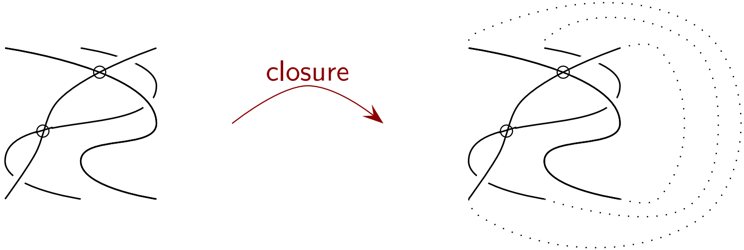
V.V. Vershinin, '01:

Virtual braids $\stackrel{Def.}{=}$ Diagrams / RII,RIII,RII^v,RIII^v,RIII^m

Virtual and mixed moves:

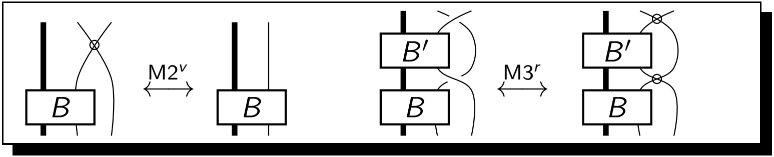


Virtual Alexander-Markov theorem



Theorem (S. Kamada, '07('00))

- Surjectivity.
- Kernel: moves M1, M2, M2^v, M3^r, M3^l.



Braids in Algebra

The **braid group** on n strands B_n:

- generators: $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$;
- relations: $\sigma_i \sigma_j = \sigma_j \sigma_i, \qquad |i-j| > 1 \qquad (Comm)$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \qquad (YBE)$$

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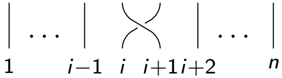
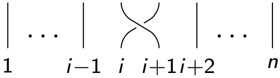
$\varphi : B_n \xrightarrow{\sim}$ braids on n strands

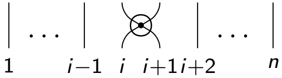
$\sigma_i \longmapsto$



Braids in Algebra

$\varphi : VB_n \xrightarrow{\sim}$ virtual braids on n strands

$\sigma_i \longmapsto$ 

$\zeta_i \longmapsto$ 

Braids in Algebra

The **virtual braid group** on n strands VB_n:

- generators: $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}; \zeta_1, \ldots, \zeta_{n-1};$
- relations:

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \qquad |i-j|>1 \qquad (Comm)$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad (YBE)$$

$$\zeta_i^2 = 1 \qquad (Inv^v)$$

$$\zeta_i\zeta_j = \zeta_j\zeta_i, \qquad |i-j|>1 \qquad (Comm^v)$$

$$\zeta_i\zeta_{i+1}\zeta_i = \zeta_{i+1}\zeta_i\zeta_{i+1} \qquad (YBE^v)$$

$$\sigma_i\zeta_j = \zeta_j\sigma_i, \qquad |i-j|>1 \qquad (Comm^m)$$

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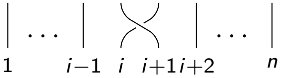
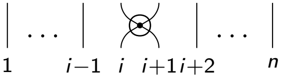
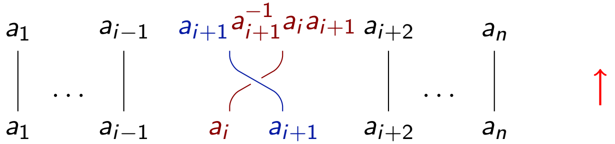
$\sigma_i \longmapsto$ 

$\zeta_i \longmapsto$ 

Braids and Representation Theory

Hurwitz action: $B_n \curvearrowright X^{\times n}$, X is a group

$\sigma_i \longmapsto$



(cf. Wirtinger presentation of $\pi_1((\mathbb{R}^2 \times [0,1]) \backslash \beta)$).

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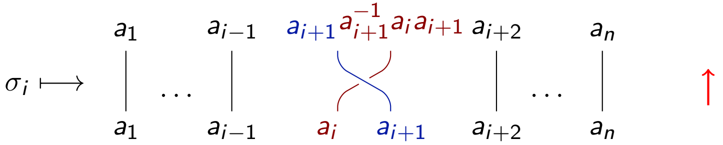

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E. Artin, '47: $\boxed{B_n \hookrightarrow \mathrm{Aut}(F_n{}^{\times n})}$, $F_n =$ the free group on n elements.

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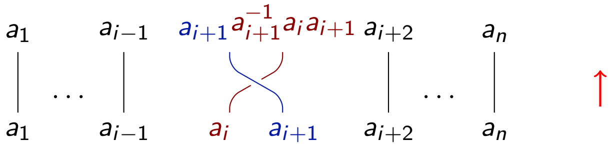

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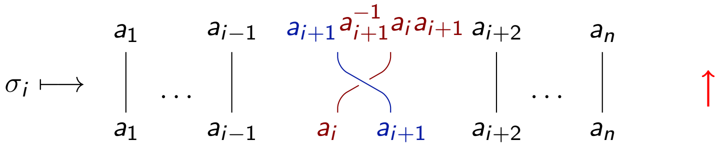

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D. Joyce, S.V. Matveev, '82: $B_n \curvearrowright X^{\times n}$, $(X, \lhd)$ is a **rack**:

$$(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c), \qquad (SD)$$
$$(a \mathbin{\widetilde{\lhd}} b) \lhd b = a = (a \lhd b) \mathbin{\widetilde{\lhd}} b.$$

Braids and Representation Theory

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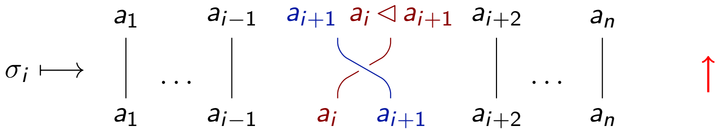

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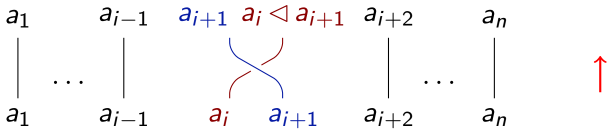

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Example: group X, $a \lhd b = b^{-1}ab$, $a \mathbin{\widetilde{\lhd}} b = bab^{-1}$.

Virtual braids and Representation Theory

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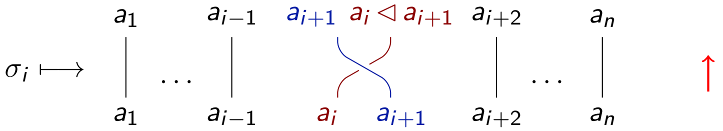

Question: $VB_n \curvearrowright X^{\times n}$?

Virtual braids and Representation Theory

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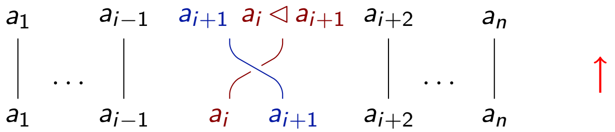
$\sigma_i \longmapsto$ 

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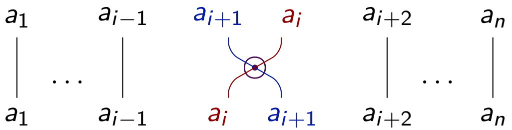
Answers:

1. $\zeta_i \longmapsto$ 

Virtual braids and Representation Theory

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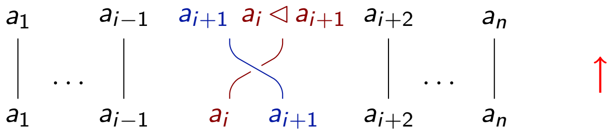
$\sigma_i \longmapsto$ 

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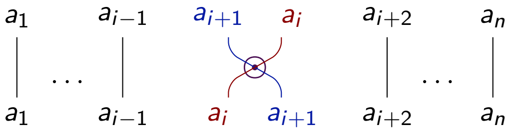
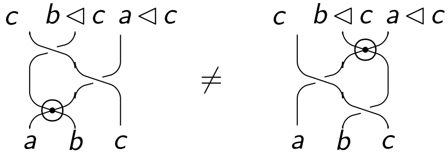
1. $\zeta_i \longmapsto$ 

Problem:

forbidden move



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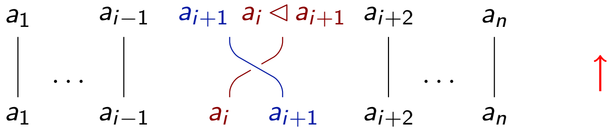

Question: $VB_n \curvearrowright X^{\times n}$?

Answers (V.O. Manturov, '02):

Virtual rack $=$ rack $(X, \lhd)$ & $f \in \mathrm{Aut}(X)$:

- f is invertible;
- $f(a \lhd b) = f(a) \lhd f(b)$.

Virtual braids and Representation Theory

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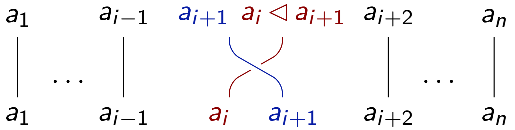

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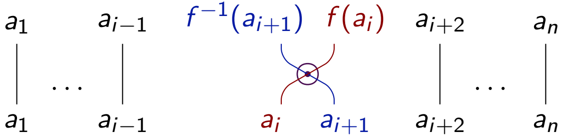
2 $\zeta_i \longmapsto$



Virtual racks

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Examples:

0. Any rack $(X, \lhd)$, $f = \mathrm{Id}$.

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Examples:

0. Any rack $(X, \lhd)$, $f = \mathrm{Id}$.
1. Any rack $(X, \lhd)$, any $a \in X$, $f : x \mapsto x \lhd a$.

Virtual racks

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   Evidence:
   - OK for $n = 2$;
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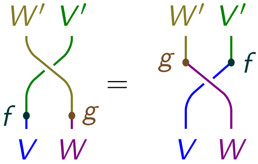

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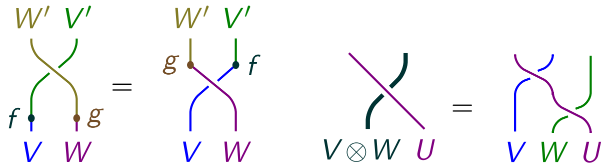

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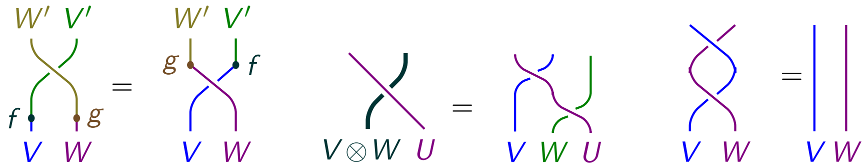
- $\mathcal{C}$ is **symmetric** if moreover $\quad c_{W,V} \circ c_{V,W} = \mathrm{Id}_{V \otimes W}$.



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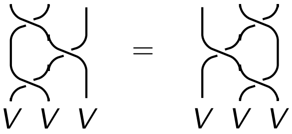

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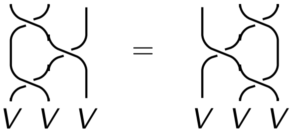

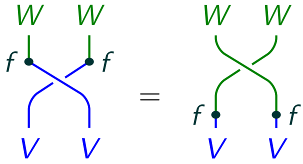

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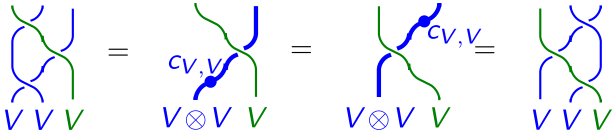
- Braided category $(\mathcal{C}, c_{\bullet,\bullet}) \rightsquigarrow$ braided objects $(V, \sigma_V := c_{V,V})$.



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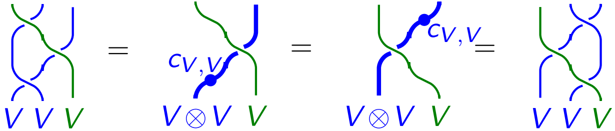

- Rack $(X, \lhd) \rightsquigarrow$ braided object $(X, (a,b) \mapsto (b, a \lhd b))$ in $(\mathbf{Set}, \times)$.

Braids and Category Theory

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Denote by $\mathcal{B}_1$ the free braided category generated by an object V. Then

$$\boxed{B_n \xrightarrow{\sim} \mathrm{End}_{\mathcal{B}_1}(V^{\otimes n})}$$

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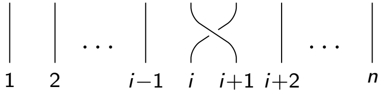

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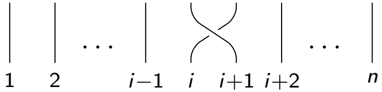

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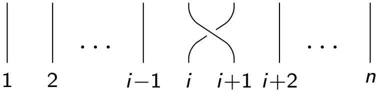

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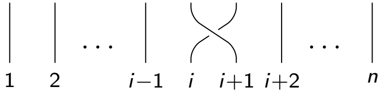

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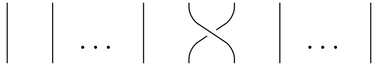

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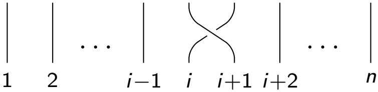

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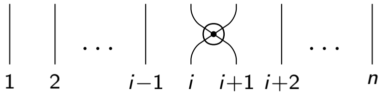

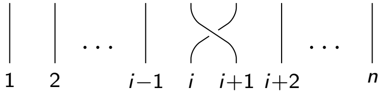

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⚠ A flexible construction $\rightsquigarrow$ **applications** to representations.

Virtual braids as hom-sets: application 1

Proposition (virtualized braidings)

Take a symmetric category $(\mathcal{C}, c_{\bullet,\bullet})$, $V \in \mathrm{Ob}(\mathcal{C})$, and $f \in \mathrm{Aut}_{\mathcal{C}}(V)$.

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$\rightsquigarrow$ a categorical interpretation of virtual racks and corr. VB_n actions.

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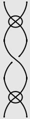
Proposition (single twisting)

Let (V, σ_V) be a braided object in a symmetric category $(\mathcal{C}, c_{\bullet,\bullet})$.
Then V admits an alternative braiding

$$\boxed{\sigma_V^c = c_{V,V} \circ \sigma_V \circ c_{V,V}}$$



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Let $\mathcal{C}$ be a monoidal category endowed with three symmetric braidings $a_{\bullet,\bullet}$, $b_{\bullet,\bullet}$, and $c_{\bullet,\bullet}$. Let (V, σ_V) be a braided object in $\mathcal{C}$. Then the map

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Corollary: Take

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$$\zeta_i \mapsto (c^{f^k}_{V,V})_i, \quad \sigma_i \mapsto (\sigma_V)_i;$$
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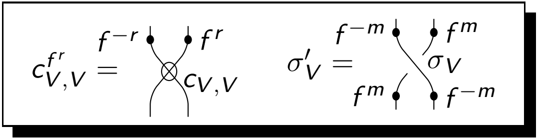

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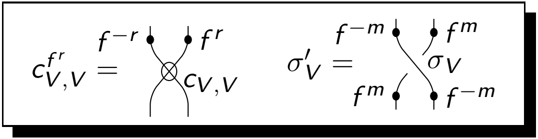

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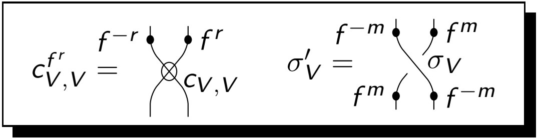

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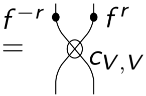
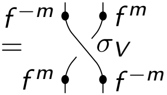
$c^{f^r}_{V,V} =$ 

$\sigma'_V =$ 

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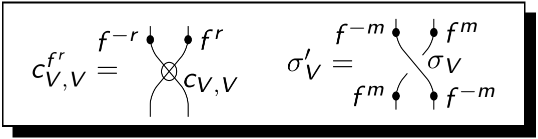

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twisted Burau representation

(Silver-Williams, '01)

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2. L., '13: include Δ in the structure of categorical rack (a local Δ).

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- Δ is coassociative central-cocommutative;
- $\lhd$ respects Δ (in the braided bialgebra sense);
- ε is a right counit;
- $\widetilde{\lhd}$ is the twisted inverse of $\lhd$.

Categorical racks: examples

category	Δ	(CSD)	cat. rack
Set	$a \mapsto (a, a)$	$(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$	❀ usual rack

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Remark: Take a linearization $\Bbbk S$ of a set S, and a linearization Δ of the map $S \ni a \mapsto (a, a)$. Suppose that $R = (\Bbbk S, \Delta, \lhd)$ is a rack in **Vect**. Then, for any $a, b \in S$, one has $a \lhd b \in S \coprod \{0\}$ (R is "almost a linearization of a usual rack").

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Conclusion:

❁ usual racks and Lie and Hopf algebras are particular cases of categorical racks;

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Vect	$v \mapsto 1 \otimes v$ $+ v \otimes 1$	$v \lhd (w \lhd u) = (v \lhd w) \lhd u$ $-(v \lhd u) \lhd w$	❀ Lie algebra

❀ Hopf algebra in **Vect**.

Remark: Vect can be replaced with any (preadditive) symmetric category.

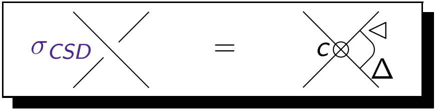
Conclusion:

❀ usual racks and Lie and Hopf algebras are particular cases of categorical racks;

❀ associative algebras are particular cases of categorical shelves.

Braiding for categorical racks

Proposition: A rack $(V, \lhd, \widetilde{\lhd}, \Delta, \varepsilon)$ in $\mathcal{C}$
$\rightsquigarrow$ a braided object (V, σ_{CSD}) in $\mathcal{C}$:



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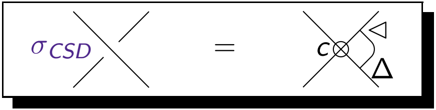

Examples:

category	Δ	alg. structure	σ_{CSD}
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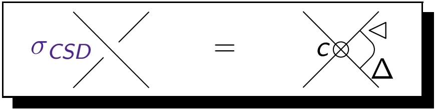

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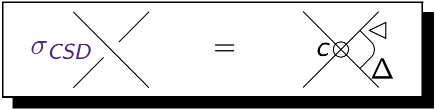

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VB_n	$\mathcal{C}$	rack in $\mathcal{C}$
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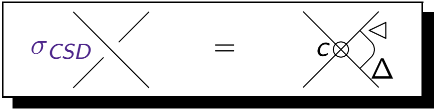

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$\rightsquigarrow$ A motivation for choosing a global Δ.

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Take a braided object (V, σ_V) in a symmetric category $\mathcal{C}$.

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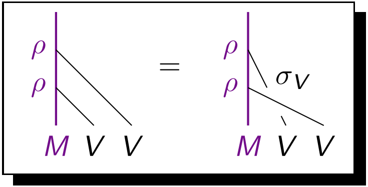

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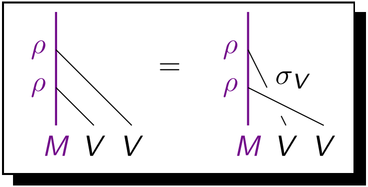
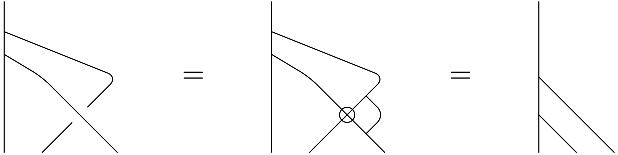


Examples: ❁ For a rack V in $\mathcal{C}$, $(V, \triangleleft)$ is a r. br. mod. over (V, σ_{CSD}).



❁ For σ_{SD}, σ_{Ass}, σ_{Lie}, one recovers usual notions of modules.

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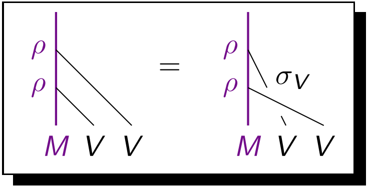
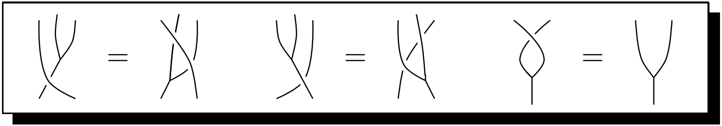


- **Coalgebra** structure for (V, σ_V): $V \overset{\Delta}{\to} V \otimes V$ s.t.



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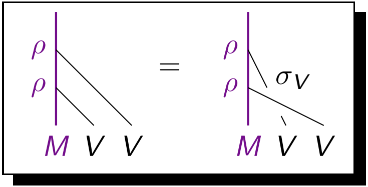

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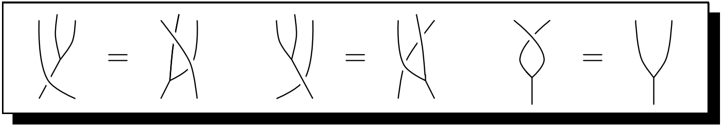


Example: for an associative algebra V, $\Delta : v \mapsto 1 \otimes v$ is a coalgebra structure for (V, σ_{Ass}).

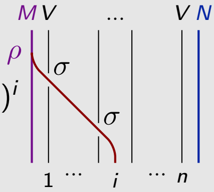
Braided homology

Theorem (L., '13)

Take a braided object (V, σ_V) in a symmetric preadditive category $\mathcal{C}$, and braided modules $(M, \rho : M \otimes V \to M)$ and $(N, \lambda : V \otimes N \to N)$ over it.

1. One has a bidifferential complex $(M \otimes T(V) \otimes N, {}^{\bullet}\delta, \delta^{\bullet})$.

$${}^{\bullet}\delta = \sum(-1)^i$$



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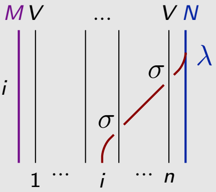

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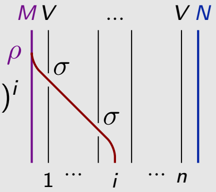

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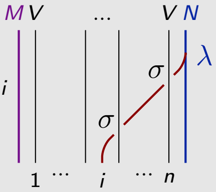

2. If Δ is a coalgebra structure for (V, σ_V), then $\sum_i s_i$ is a morphism of bidifferential complexes.

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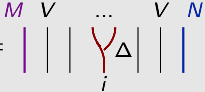

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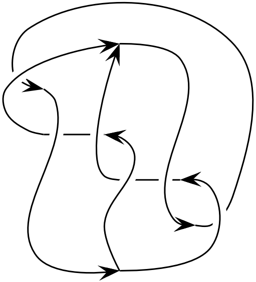
Knotted trivalent graphs

3-**graphs**



Importance:

- handlebody-knots;
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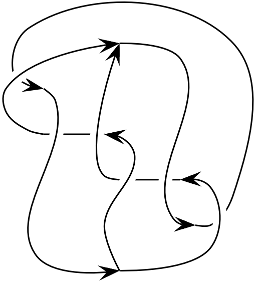

L.H. Kauffman, S. Yamada, D.N. Yetter, '89:

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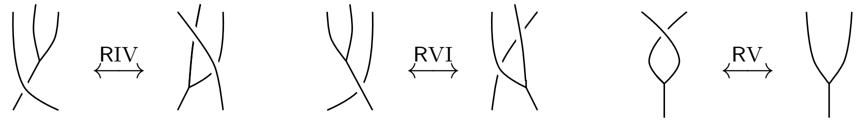
3-graphs $\cong$ Diagrams / RI-RVI

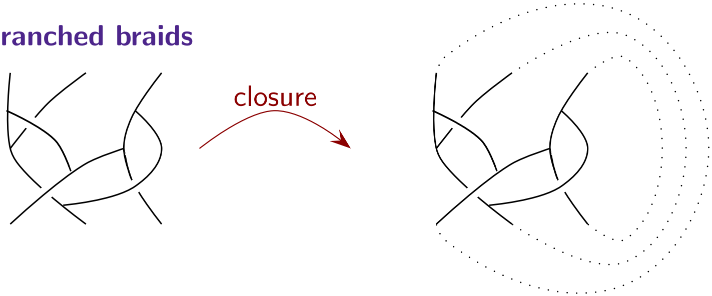


Branched braids in Topology

? $\xrightarrow{\text{closure}}$ 3-graphs

Branched braids in Topology

Branched braids



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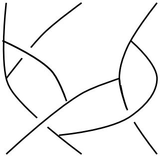
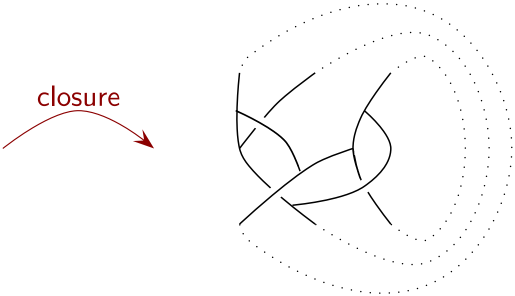
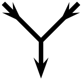
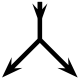




Here we work with oriented **poleless** 3-graphs:

only zip  and unzip  vertices.

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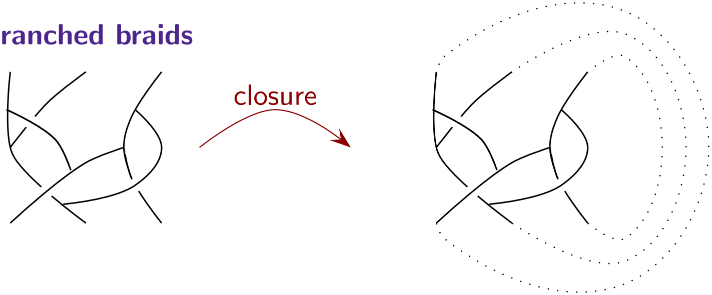

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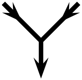
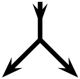
only zip  and unzip  vertices.

Branched braids $\cong$ Diagrams $/$ RII-RVI

Branched Alexander-Markov theorem

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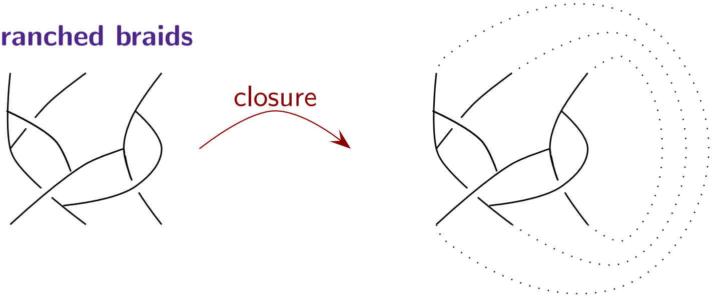

Theorem (K. Kanno - K. Taniyama, '10; S. Kamada - L., '14)

- Surjectivity.
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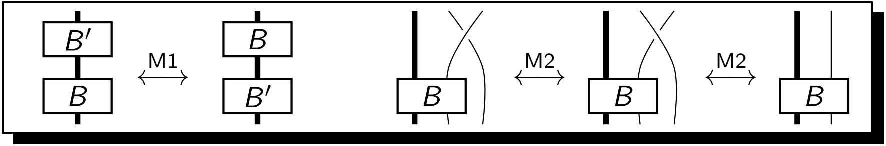

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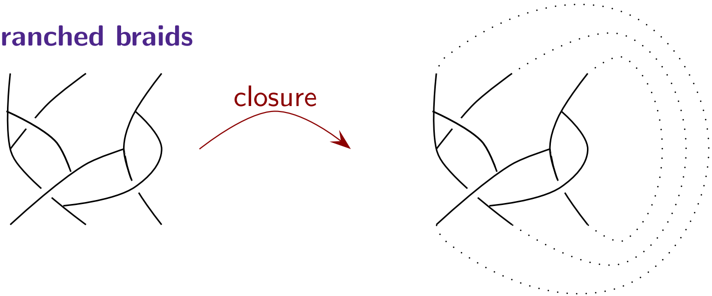

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Generalizations:

- Graph-braids (vertices of arbitrary valence).
- Virtual and welded versions.

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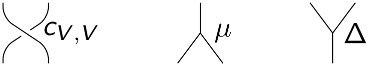

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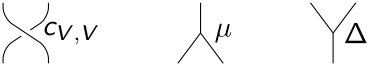

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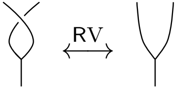

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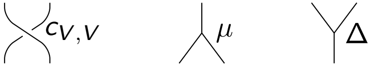

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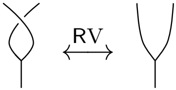

Corollary: for any commutative co-commutative algebra-coalgebra object (V, μ, Δ) in a braided category $\mathcal{C}$, one has $BB_n \curvearrowright V^{\otimes n}$.