Qualgebras: a bridge between knotted graphs and axiomatizations of groups

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OCAMI, Osaka

Topology Seminar, University of Tsukuba
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Based on:
✎ Qualgebras and Knotted 3-Valent Graphs, *ArXiv*, 2014
✎ (With S. Kamada) Alexander and Markov Theorems for Graph-Braids, *in progress*

Qualgebras: a bridge between knotted graphs and axiomatizations of groups

algebra

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algebra

topology

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Part 1:

How a Knot Theorist Would Invent Qualgebras

Knot diagrams: from illustration to manipulation

1926 K. Reidemeister:

$$\text{Knots} \quad \cong \quad \text{Diagrams} \Big/ \text{R-moves}$$

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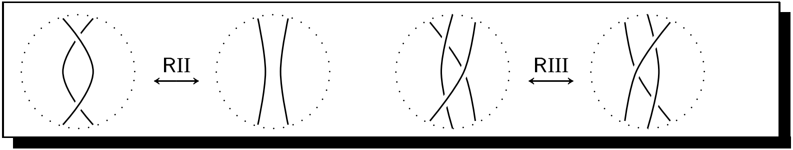
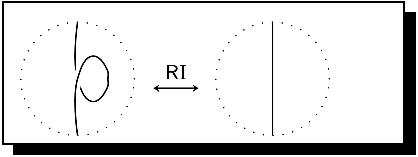
Reidemeister moves:





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Combinatorial knot invariants:

$$\begin{tikzcd}
 & \text{Diagrams} \arrow[r, mapsto] \arrow[d, two heads] & \text{something} \\
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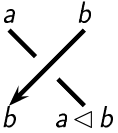
Quandles as an algebraization of knots



colorings by $(S, \triangleleft)$

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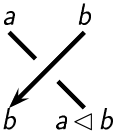

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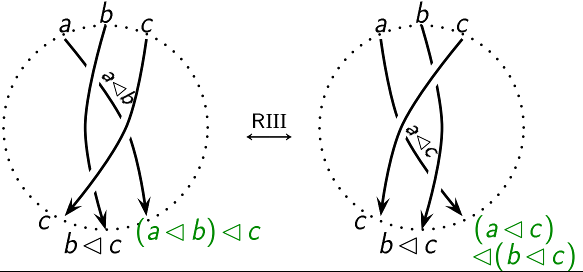

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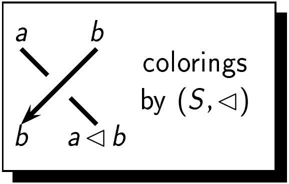

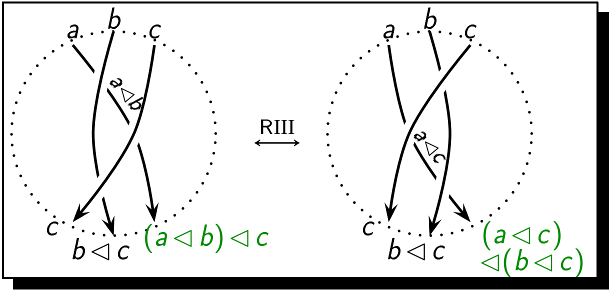

$$\boxed{\text{RIII} \quad \leftrightarrow \quad (a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c) \quad \text{(SD)}}$$

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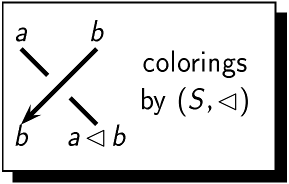

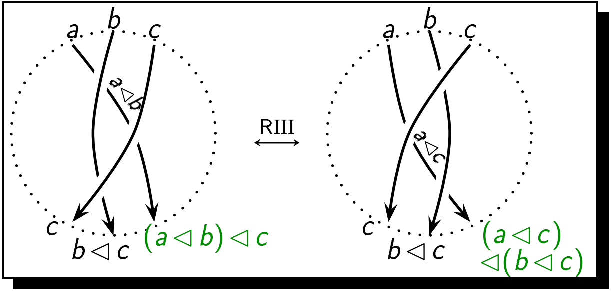

RIII	$\leftrightarrow$	$(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$	(SD)
RII	$\leftrightarrow$	$(a \lhd b) \tilde{\lhd} b = a = (a \tilde{\lhd} b) \lhd b$	(Inv)
RI	$\leftrightarrow$	$a \lhd a = a$	(Idem)

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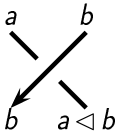

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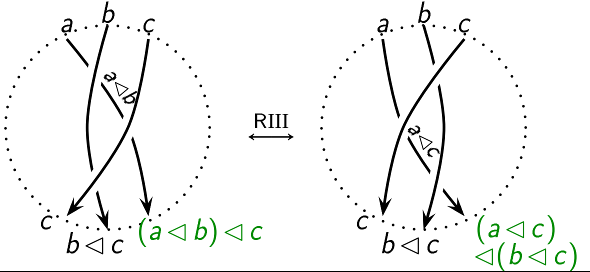

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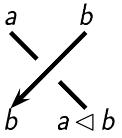

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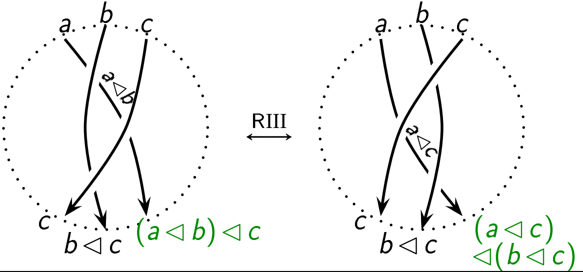

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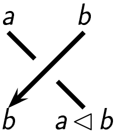

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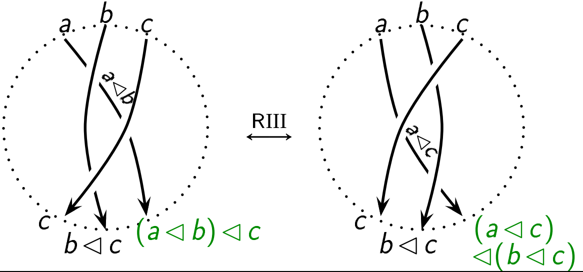

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Example

Group $G \rightsquigarrow$

$Conj(G) = (G, g \lhd h = h^{-1}gh)$.

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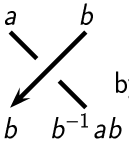

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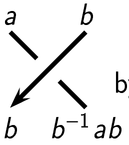

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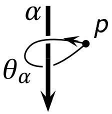
Wirtinger presentation:



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$\updownarrow$
$Rep(\pi_1(\mathbb{R}^3 \setminus K), G)$

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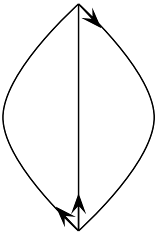
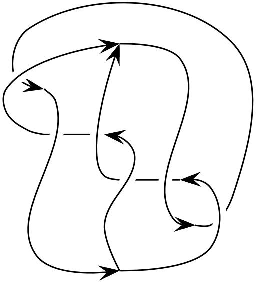
Knotted 3-valent graphs

Standard and Kinoshita-Terasaka* Θ*-curves:



$\boxed{\Theta_{st}}$



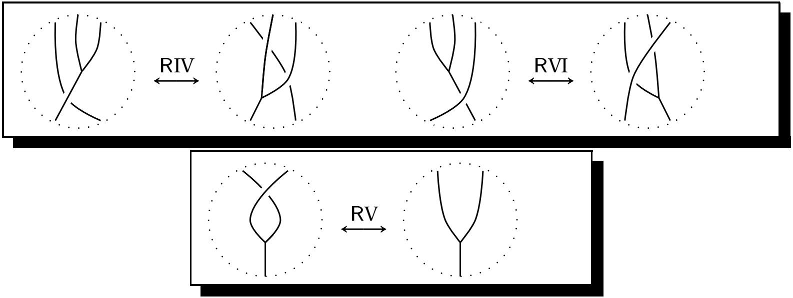
$\boxed{\Theta_{KT}}$

Importance:

- handlebody-knots;
- foams (categorification, 3-manifolds);
- form a finitely presented algebraic system (⚠ knots do not).

Reidemeister moves for 3-graphs

1989 L.H. Kauffman, S. Yamada, D.N. Yetter:



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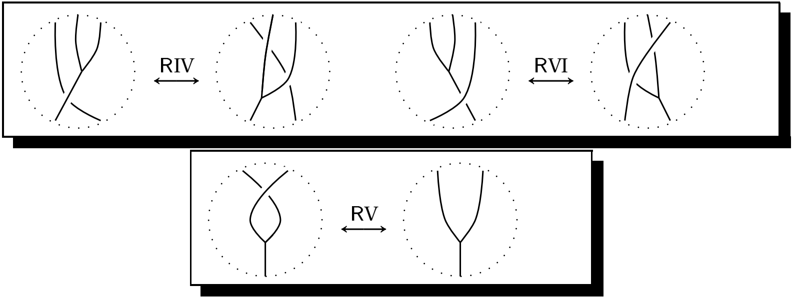

3-Graphs $\cong$ Diagrams / RI-RVI

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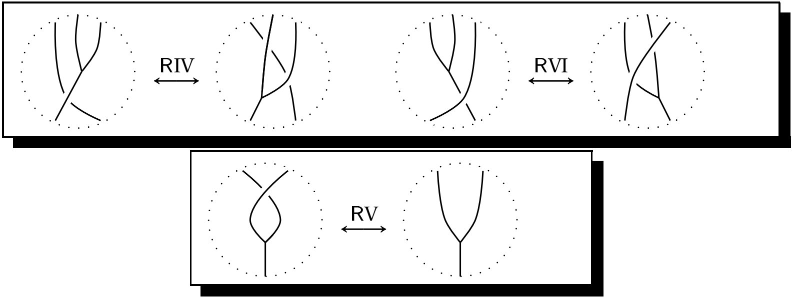

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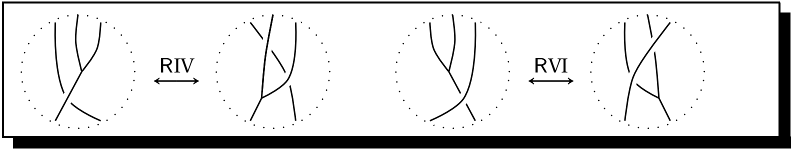

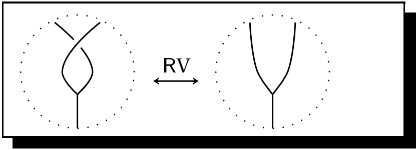

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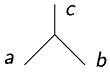
Question: How to extend quandle colorings to 3-graphs?

Quandle colorings for 3-graphs



G is a group;
$a^{\pm 1} b^{\pm 1} c^{\pm 1} = 1$

generalizations:
G-family of quandles (2012 Ishii-Iwakiri-Jang-Oshiro),
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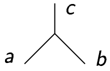
Quandle colorings for 3-graphs

- 

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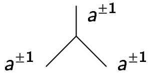
- 

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a *vertex constant version* of $\pi_1(\mathbb{R}^3 \setminus \Gamma)$
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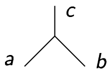
- 

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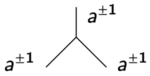
- 

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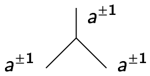
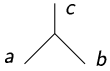
- 

S is a *symmetric* quandle; $\forall\, x,\ ((x \lhd a) \lhd a) \lhd a = x$

(2007 Fleming-Mellor)

Quandle colorings for 3-graphs

- 

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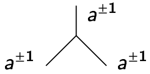
- 

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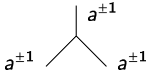
- 

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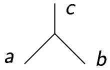
- 

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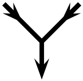
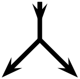
Orientation

Well-oriented 3-graphs: only ***zip***  and ***unzip***  vertices.

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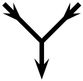
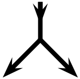
Well-oriented 3-graphs: only ***zip***  and ***unzip***  vertices.

Proposition: Every 3-graph admits a well-orientation.

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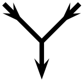
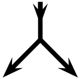
Well-oriented 3-graphs: only ***zip***  and ***unzip***  vertices.

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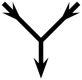
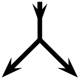
Well-oriented 3-graphs: only ***zip***  and ***unzip***  vertices.

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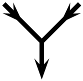
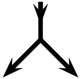
Well-oriented 3-graphs: only ***zip***  and ***unzip***  vertices.

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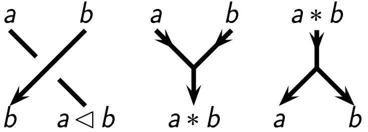
Theorem (A. Ishii): all well-orientations are connected by single-edge orientation switches.

Qualgebras as an algebraization of 3-graphs

colorings
by $(S, \lhd, *)$



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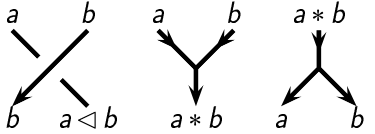
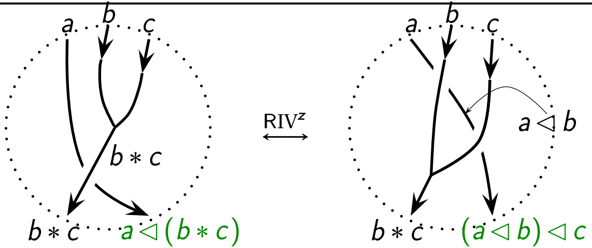
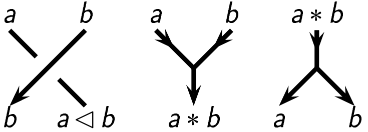
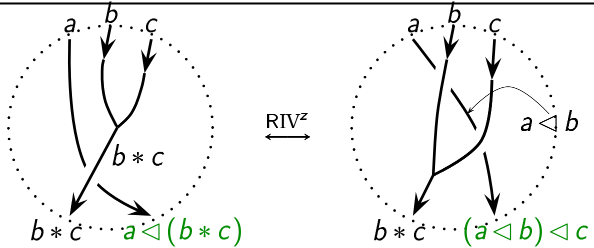




Qualgebras as an algebraization of 3-graphs

colorings by $(S, \lhd, *)$





RIV	$\leftrightarrow$	$a \lhd (b * c) = (a \lhd b) \lhd c$
RVI	$\leftrightarrow$	$(a * b) \lhd c = (a * c) \lhd (b * c)$
RV	$\leftrightarrow$	$a * b = b * (a \lhd b)$

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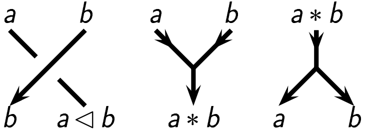

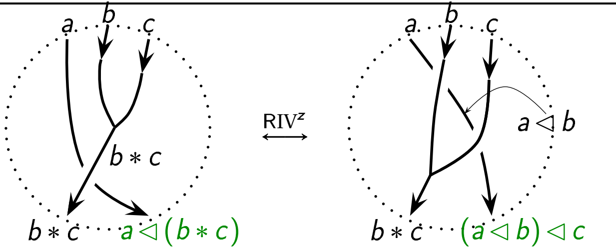

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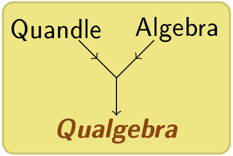

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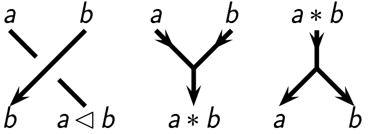

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Quandle Algebra

↓

Qualgebra

3-graph invariants $\overset{\text{colorings}}{\leftsquigarrow}$ qualgebra

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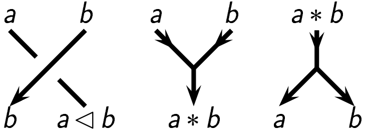

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Quandle Algebra

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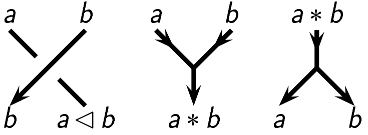
Example

$$\text{Group } G \rightsquigarrow QA(G) = (G, g \lhd h = h^{-1}gh, g * h = gh).$$

Qualgebras as an algebraization of 3-graphs

colorings by $(S, \triangleleft, *)$



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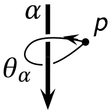
Wirtinger presentation:



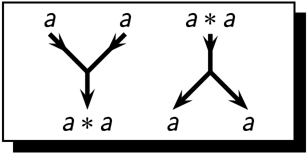
colorings by $QA(G)$

$\updownarrow$

$Rep(\pi_1(\mathbb{R}^3 \setminus \Gamma), G)$

Computation example

Isosceles colorings:



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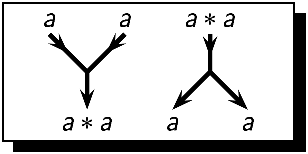
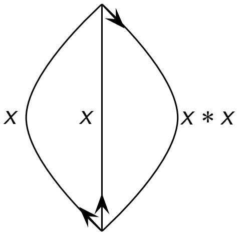
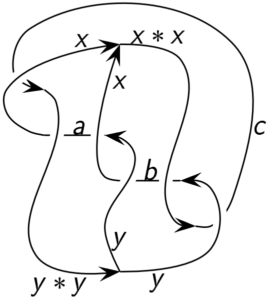


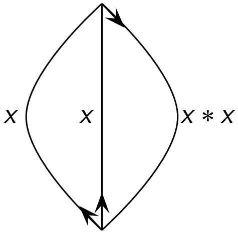


Θ_{st}

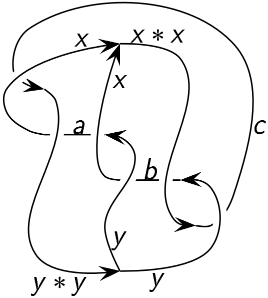


Θ_{KT}

Computation example



$\boxed{\Theta_{st}}$



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$$(\star)\begin{cases} a = x \lhd (y * y) = y \lhd x, \\ b = x \widetilde{\lhd} y = y \widetilde{\lhd} (x * x), \\ c = (y * y) \lhd x = (x * x) \widetilde{\lhd} y. \end{cases}$$

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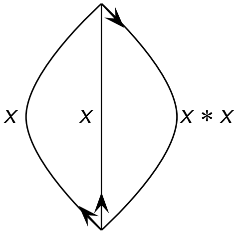

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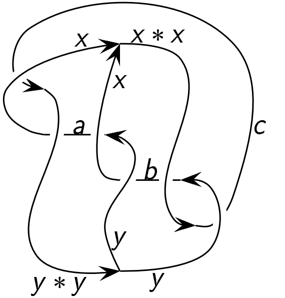

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Qualgebra $(S_4, g \triangleleft h = h^{-1}gh, g * h = gh)$.

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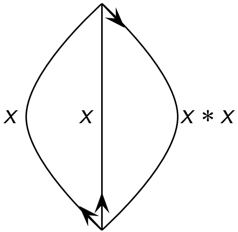

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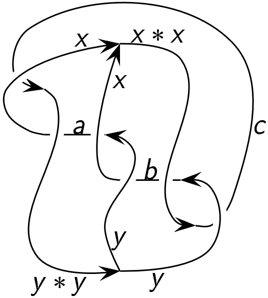

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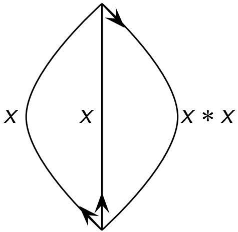

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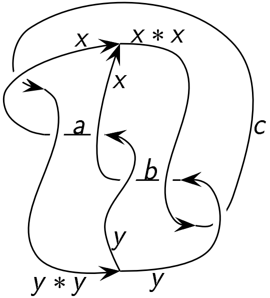

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$$\#\mathcal{C}ol^{iso}_{S_4}(\Theta_{st}) = S_4 \qquad \#\mathcal{C}ol^{iso}_{S_4}(\Theta_{KT}) > \#S_4$$

Part 2:

How an Algebraist Would Invent Qualgebras

Group qualgebras

Example 1

Group G $\rightsquigarrow$ ***group qualgebra*** $QA(G) = (G, g \lhd h = h^{-1}gh, g * h = gh)$.

Group qualgebras

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abstract level	quandle axioms	specific qualgebra axioms
topology	moves RI-RIII	moves RIV-RVI
groups	conjugation	conjugation-multiplication interaction

Group qualgebras

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Quandle axioms $\Rightarrow$ all properties of conjugation.

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Quandle axioms $\Rightarrow$ all properties of conjugation.

⚠ Qualgebra axioms $\not\Rightarrow$ all properties of conjugation/multiplication interaction.

$$(b \lhd a) * (a \lhd b) = ((a \widetilde{\lhd} b) \lhd a) * b$$

in $QA(G)$: $\quad a^{-1}bab^{-1}ab = a^{-1}bab^{-1}ab$

in free qualgebras: $\quad$ false

Other qualgebra examples

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Example 1′

Group G & $X \subset G \rightsquigarrow$ the sub-qualgebra of $QA(G)$ generated by X.

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Trivial qualgebra $(S, a \triangleleft b = a, a * b)$, where $*$ is commutative.

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$\rightsquigarrow$ Abstract graph invariants.

An infinite family of exotic examples

Example S_n

Set X & commutative operation $\star$ & zero element 0 ($0 \star x = x \star 0 = 0$)

An infinite family of exotic examples

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$$Q_{X,n} = \{((x_1,\dots,x_n),g) \in X^{\times n} \times S_n \,|\, x_i = x_j = 0 \text{ whenever } g(i) = j \text{ with } i \neq j\},$$

$$(\overline{x},g) \lhd (\overline{y},h) = (\overline{x} \cdot h, h^{-1}gh),$$

$$(\overline{x},g) * (\overline{y},h) = (\overline{x} \star \overline{y}, gh)$$

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$$\rightsquigarrow \qquad \text{two 5-element qualgebras.}$$

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$\rightsquigarrow$ two 5-element qualgebras.

- Commutative.
- Associative.
- Not cancellative $\Rightarrow$ do not come from groups:

$$((0,0),\tau) *_i ((x_1,x_2),\mathrm{Id}) = ((0,0),\tau), \qquad i = 1,2.$$

Towards a classification

Example 3

Size $\leqslant 3$: only trivial qualgebras.

Towards a classification

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Non-trivial qualgebra structures on $Q = \{p, q, r, s\}$:

$$\begin{aligned} &\text{put } \overline{p} = q, \overline{q} = p, \overline{r} = r, \overline{s} = s; \\ &x \lhd r = \overline{x}, \qquad x \lhd y = x \text{ for other } y; \\ &* \text{ is commutative,} \qquad \overline{x} * \overline{y} = \overline{x * y}, \\ &r * x = r \text{ for } x \neq r, \\ &r * r = s * s = p * q = s, \qquad p * p, p * s \in \{p, q, s\} \end{aligned}$$

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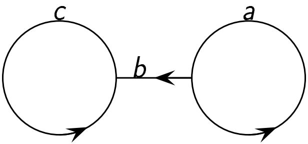
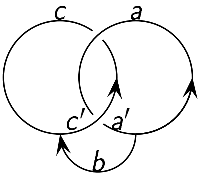
Question: Continue the classification.

Computation example

Standard and Hopf cuff graphs:



$\boxed{C_{st}}$



$\boxed{C_H}$

$$a' = ((a \widetilde{\lhd} c) \lhd a$$
$$c' = c \lhd a$$

$$\#\mathcal{Col}_Q(C_{st}) = \#\{(a,b,c) \in Q \,|\, b * a = a,\ b * c = c\} = 18,$$
$$\#\mathcal{Col}_Q(C_H) = \#\{(a,b,c) \in Q \,|\, b * a = a \lhd c,\ b * c = c \lhd a\} = 14.$$

Qualgebras with inversion

Question: How far qualgebras are from groups?

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Group $G \rightsquigarrow QA^*(G) = (G, g \lhd h = h^{-1}gh, g * h = gh, \rho(h) = h^{-1})$.

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Together these give a ***Symmetric qualgebra***.

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- Maps $a \mapsto a * b$ and $a \mapsto b * a$ are bijections
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The first three conditions: ***Symmetric quandle*** (1996 S. Kamada)

All four conditions: ***Symmetric qualgebra***

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- ρ is defined uniquely.

Topological motivation

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Symmetric quandle (1996 S. Kamada) $\Big\}$ ***Symmetric qualgebra***

abstract level	good involution axioms
topology	unoriented 3-graphs
groups	conjugation- and multiplication-inversion interactions

Symmetric qualgebras: examples

Example 0

Symmetric trivial qualgebras $\longleftrightarrow$ Latin squares which

- are symmetric w.r.t. the main diagonal, and
- contain a row $\sigma \in Bij(S)$ iff contain a row σ^{-1}.

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Example 3

$S = \{x, y, z\}$, $a \lhd b = a$, $*$ is commutative.

$*$	x	y	z
x	x	y	z
y	y	z	x
z	z	x	y
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$\mathbb{Z}/3\mathbb{Z}$

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- Trivial qualgebras:

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$\leftarrow QA(\mathbb{Z}/4\mathbb{Z})$

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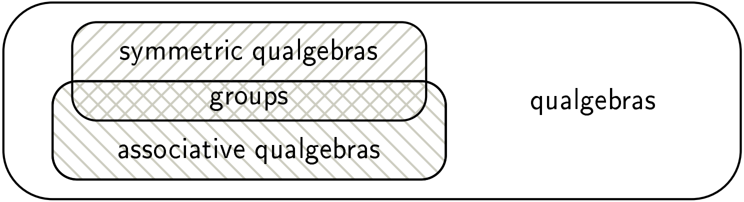

From quandles to qualgebras: theory

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That is, $T_{b \lhd c} = T_c^{-1} T_b T_c$.

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Question: A good qualgebraizability criterion?

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Non-uniqueness

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$QA(S_n)$ is the unique qualgebraization of $Conj(S_n)$ $(\Longleftarrow$ T is injective).

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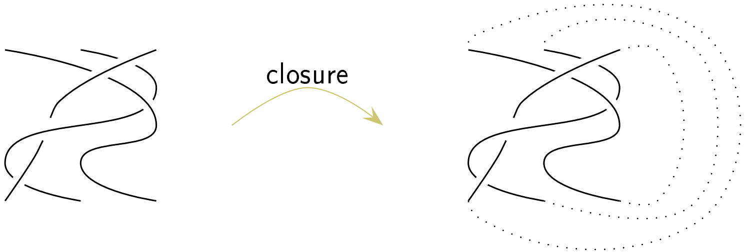
Part 3:

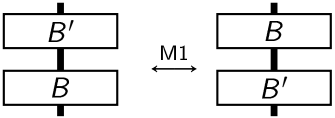
Variations of Qualgebra Ideas

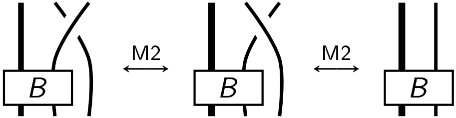
Alexander-Markov theorem



Theorem (1923 Alexander; 1935 Markov)

- Surjectivity.
- Kernel: moves M1 and M2.





Braid and knot invariants

Knot invariants out of braid invariants:

$$\begin{tikzcd} & \text{Braids} \arrow[r, mapsto] \arrow[d, two heads] & \text{something} \\ \text{Knots} \cong & \text{Braids} \big/ {\text{M1,M2}} \arrow[ur, dashed] & \end{tikzcd}$$

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Knots $\cong$ Braids $\Big/ _{\text{M1,M2}}$

1925 E. Artin: $\boxed{\text{Braids} \cong \text{Diagrams} \Big/ _{\text{RII-RIII}}}$

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Many knot invariants adapt to the braid case, with possible enhancements.

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- Weaker structure: ***rack*** ($=$ quandle without $a \lhd a = a$) $\Longleftarrow$ no RI.
- Operator invariants instead of counting invariants:

braids with n strands $\longrightarrow \mathrm{Aut}(S^{\times n})$,

$\beta \longmapsto$ (colors of upper arcs $\mapsto$ colors of lower arcs).

Branched Alexander-Markov theorem

Question: : A closure procedure for 3-graphs?

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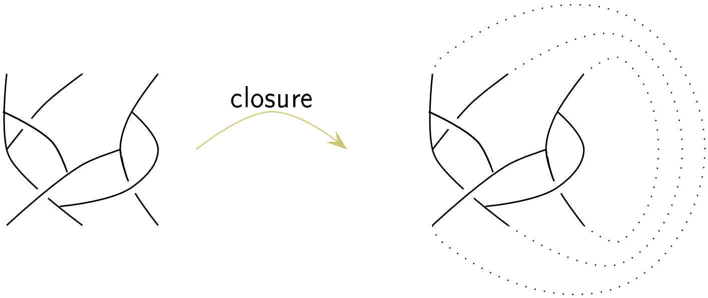
Branched braids



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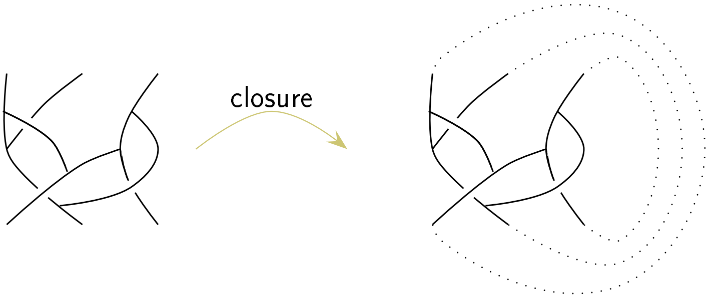

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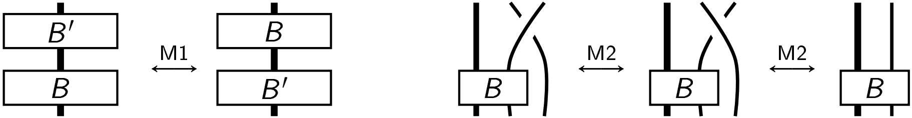

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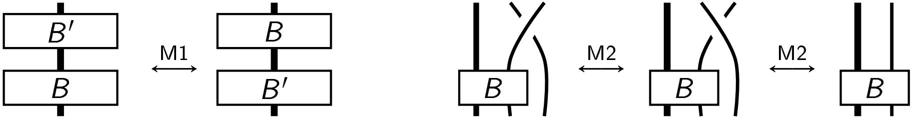
branched braids $\xrightarrow{\text{closure}}$ 3-graphs

Theorem (2010 K. Kanno - K. Taniyama; 2014 S. Kamada - L.)

- Surjectivity.
- Kernel: moves M1 and M2.



Generalizations

- Graph-braids (vertices of arbitrary valence).
- Virtual and welded versions.

Branched braid and 3-graph invariants

3-Graph invariants out of B-braid invariants:

$$\begin{tikzcd} & B\text{-braids} \arrow[r, mapsto] \arrow[d, two heads] & \text{something} \\ \text{3-Graphs} \cong & B\text{-braids} \big/ {\text{M1,M2}} \arrow[ur, dashed] & \end{tikzcd}$$

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B-braid invariants $\overset{\text{colorings}}{\rightsquigarrow}$ ***weak qualgebra*** (omit $a \lhd a = a$)

Getting more out of qualgebra colorings

$$\begin{array}{rccc} \text{diagrams:} & D & \overset{\text{R-move}}{\rightsquigarrow} & D', \\ \text{colorings:} & \mathscr{C} & \rightsquigarrow & \mathscr{C}', \\ \text{coloring sets:} & \mathcal{Col}_S(D) & \overset{\text{bij}}{\longleftrightarrow} & \mathcal{Col}_S(D'). \end{array}$$

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$$\omega(\mathscr{C}) = \omega(\mathscr{C}'),$$

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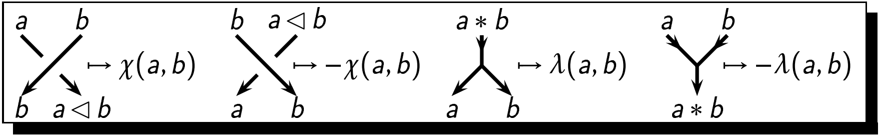
Inspiration: *Quandle cocycle invariants* of knots (1999 Carter-Jelsovsky-Kamada-Langford-Saito).

Qualgebra cocycle invariants for 3-graphs



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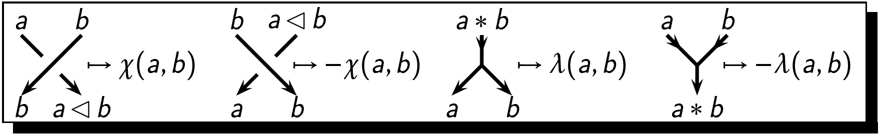

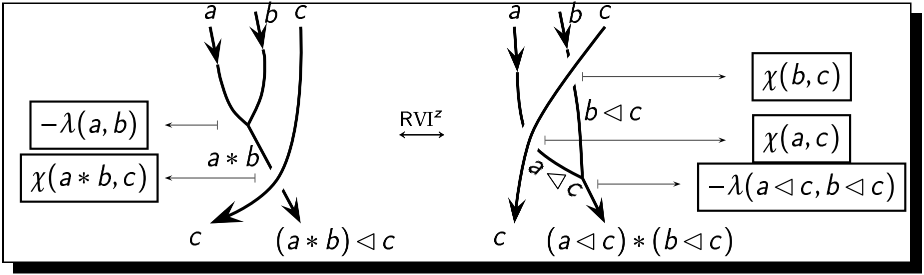

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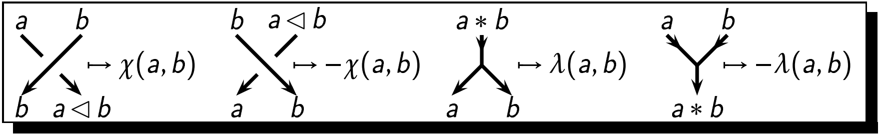

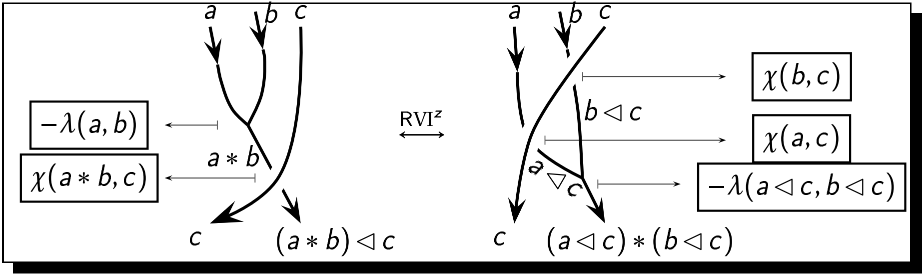

RIV $\leftrightarrow$ $\chi(a, b * c) = \chi(a, b) + \chi(a \lhd b, c)$

RVI $\leftrightarrow$ $\chi(a * b, c) + \lambda(a \lhd c, b \lhd c) = \chi(a, c) + \chi(b, c) + \lambda(a, b)$

RV $\leftrightarrow$ $\chi(a, b) + \lambda(a, b) = \lambda(b, a \lhd b)$

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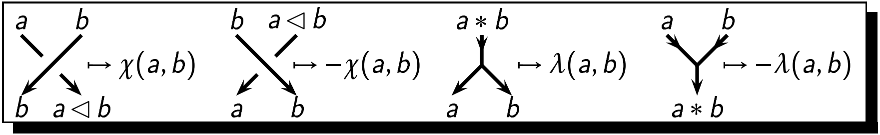

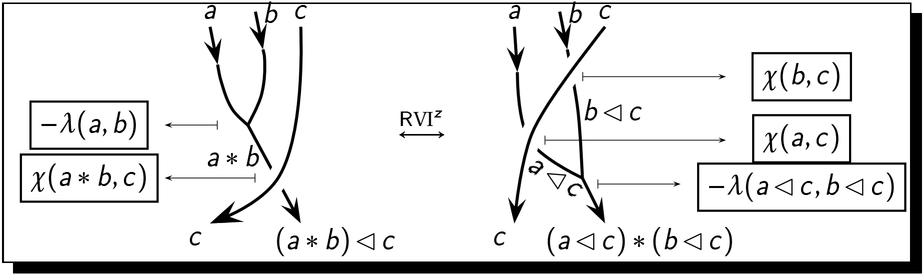

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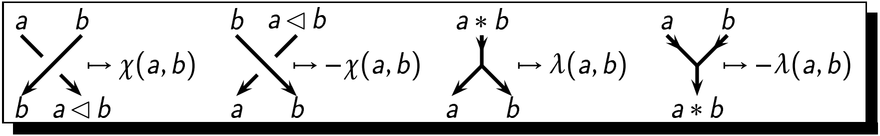
Qualgebra 2-***cocycle***

RI-RIII are automatic.

Qualgebra cocycle invariants for 3-graphs



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3-graph invariants $\overset{\text{colorings}}{\underset{\text{weights}}{\leftrightsquigarrow}}$ qualgebra & 2- or 3-cocycle

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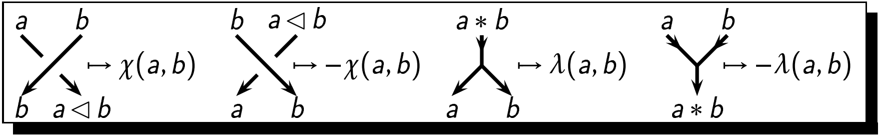

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Qualgebra cocycle invariants $\supseteq$ qualgebra counting invariants.

Towards qualgebra cohomology

Qualgebra 2-***coboundaries***:

$$\phi : S \to \mathbb{Z} \quad \rightsquigarrow \quad \chi(a,b) = \phi(a \lhd b) - \phi(a), \qquad \rightsquigarrow \quad \text{trivial}$$

$$\lambda(a,b) = \phi(a) + \phi(b) - \phi(a * b) \qquad \text{graph invariants}$$

Towards qualgebra cohomology

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Question: : Define and study qualgebra cohomology in higher degrees?

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Cf. the cohomology of multiple conjugation quandles (Carter-Ishii-Saito-Yang).

Towards qualgebra cohomology

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Enhancements

- Region coloring and shadow cocycle invariants $\leftsquigarrow$ qualgebra 3-cocycles.

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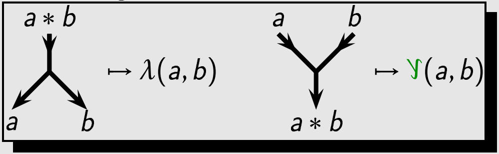
Enhancements

- Region coloring and shadow cocycle invariants $\leftsquigarrow$ qualgebra 3-cocycles.
- Distinguish zip- and unzip-vertices:



Qualgebra cocycles: example

Example 4

$$Q = \{p, q, r, s\} \qquad \overline{p} = q, \overline{q} = p, \overline{r} = r, \overline{s} = s;$$

$$x \lhd r = \overline{x}, \qquad x \lhd y = x \text{ for other } y;$$

$$* \text{ is commutative}, \qquad \overline{x} * \overline{y} = \overline{x * y},$$

$$r * x = r \text{ for } x \neq r,$$

$$r * r = s * s = p * q = s, \qquad p * p, p * s \in \{p, q, s\}$$

- $Z^2(Q) \cong \mathbb{Z}^8$
- $B^2(Q) \cong \mathbb{Z}^4$
- $H^2(Q) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^4$

どうもありがとう