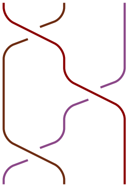
New aspects of the Yang-Baxter equation

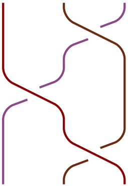
Victoria LEBED

Jean Leray Mathematics Institute, University of Nantes

Symposium on Mathematical Physics
November 10, 2014



$\longleftrightarrow$



1 Yang-Baxter equation

✓ A vector space V (or an object in any monoidal category)

✓ $\sigma\colon V^{\otimes 2} \to V^{\otimes 2}$

Yang-Baxter equation (YBE):

$$\boxed{\sigma_1 \circ \sigma_2 \circ \sigma_1 = \sigma_2 \circ \sigma_1 \circ \sigma_2 \colon V^{\otimes 3} \to V^{\otimes 3}}$$

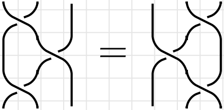
where $\sigma_i = \mathrm{Id}_V^{\otimes i-1} \otimes \sigma \otimes \mathrm{Id}_V^{\otimes \cdots}$.

A map σ satisfying YBE is a braiding.

$V \otimes V$

$\sigma \longleftrightarrow$ 

$V \otimes V$



(Reidemeister III)

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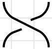
$\sigma \longleftrightarrow$  $\uparrow$

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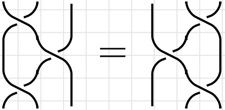
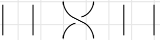


(Reidemeister III)

rep. of B_n^+ (pos. braid monoid):

$i \; i+1 \quad n$

 $\mapsto \sigma_i$

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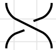
$\sigma \longleftrightarrow$  $\uparrow$

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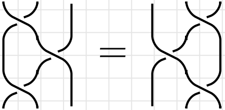

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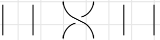
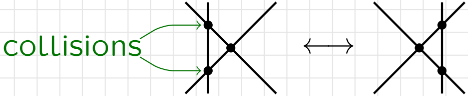
 $\mapsto \sigma_i$

rep. of B_n (braid group)

 $\mapsto \sigma_i^{-1}$

2 YBE in physics

✓ Particle physics: factorization condition for the dispersion matrix in the 1-dim. n-body problem (McGuire, Yang, 60').



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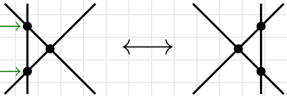
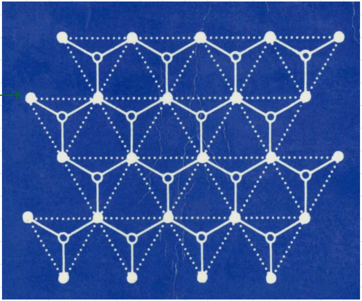
collisions



✓ Statistical mechanics: partition function for exactly solvable lattice models (Onsager, 1944, Ising model; Baxter, 70', 8-vertex, hard hexagon & chiral Potts models).

Boltzmann weights



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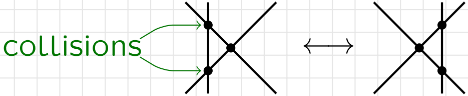

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✓ Quantum inverse scattering method for completely integrable systems (Faddeev et al., 1979).

✓ Factorizable S-matrices in 2-dim. quantum field theory (Zamolodchikov, 1979).

✓ Quantum group (Drinfel′d, 80').

✓ C^* algebras (Woronowicz, 80').

✓ Conformal field theory.

3 A homology theory for the YBE

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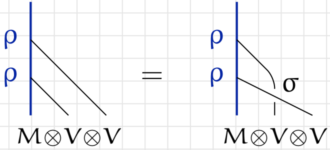
Ingredients:

✓ A braided vector space (V, σ);

✓ a left braided V-module: $(M, \rho \colon M \otimes V \to M)$ s.t.

$$\rho \circ \rho_1 = \rho \circ \rho_1 \circ \sigma_2 : \\ M \otimes V \otimes V \to M$$



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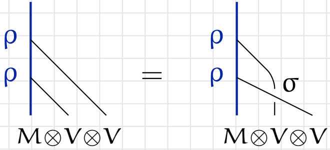

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Theorem (L. 2013): $M \otimes T(V) \otimes N$ carries a family of differentials $\delta^{(\alpha,\beta)} = \alpha\, {}^{\bullet}\delta + \beta\, \delta^{\bullet}$, $\alpha, \beta \in \mathbb{k}$.

(I.e., $\delta^{(\alpha,\beta)} \circ \delta^{(\alpha,\beta)} = 0$.)

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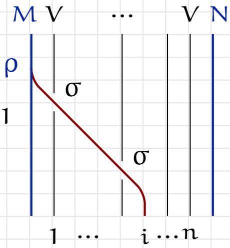
$${}^\bullet\delta = \sum (-1)^{i-1}$$



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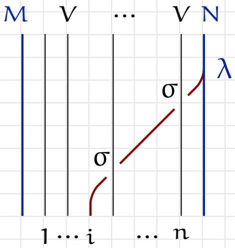

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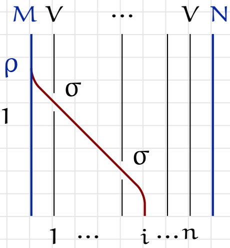
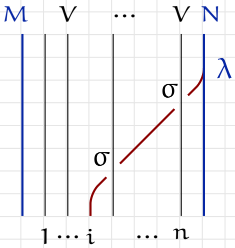
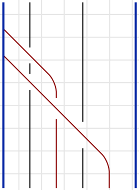
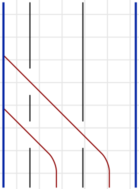
${}^{\bullet}\delta = \sum (-1)^{i-1}$ 

$\delta^{\bullet} = \sum (-1)^{i-1}$ 

Proof:

 $\overset{\text{YBE}}{=}$  $\overset{\text{br. mod.}}{=}$  & sign $= (-1)^{\#\text{cross.}}$

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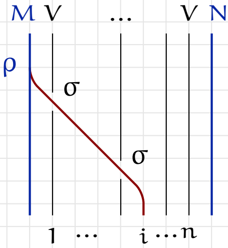
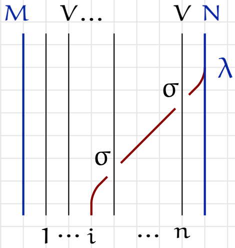
${}^{\bullet}\delta = \sum (-1)^i$ 

$\delta^{\bullet} = \sum (-1)^i$ 

<u>Remarks</u>:

✓ Functoriality.

✓ Interpretation in terms of quantum shuffles (Rosso, 1995).

✓ Duality $\rightsquigarrow$ a cohomology theory.

✓ Pre-cubical structure.

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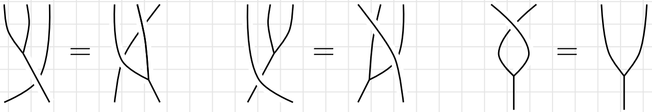
Braided coalgebra: br. v. sp. (V, σ) & $\Delta : V \to V \otimes V$ s.t.



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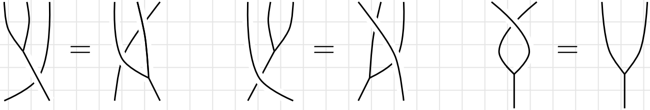

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Theorem (L. 2013): All $\delta^{(\alpha,\beta)}$ restrict to $\sum_i \mathrm{Im}(\Delta_i)$.

$\rightsquigarrow$ normalization

4 Alg. structures via braidings

A Associative algebras

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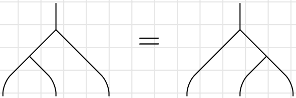
(A) Associative algebras

$(V, \mu\colon V \otimes V \to V, \xi\colon \Bbbk \to V)$, $\xi(\alpha) = \alpha 1_V$, s.t.

Associativity:
$\mu \circ \mu_1 = \mu \circ \mu_2$



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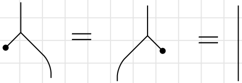

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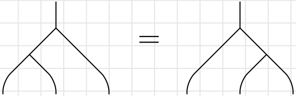

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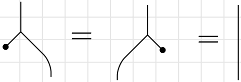
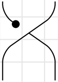


$\rightsquigarrow$ "Associativity braiding"

$\sigma_{Ass} = \xi\otimes\mu\colon \boxed{v\otimes w \mapsto 1\otimes v\cdot w}$



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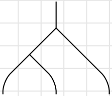
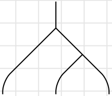
 $=$ 

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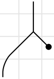
 $=$  $=$

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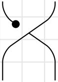

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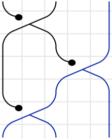
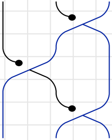
 $=$ 

4 Alg. structures via braidings

A Associative algebras

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✓ Braided homologies for (V,σ_{Ass}) include

→ bar differential; → Hochschild; → group hom.

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Ⓑ Leibniz algebras

$(V, \mu\colon V\otimes V \to V, \xi\colon \Bbbk \to V)$, $\xi(\alpha) = \alpha 1_V$, s.t.

Leibniz identity: $\mu\circ\mu_2 = \mu\circ\mu_1 - \mu\circ\mu_1\circ\tau$, where $\tau\colon w\otimes u \to u\otimes w$

$$[v,[w,u]] = [[v,w],u] - [[v,u],w]$$

Lie unit axiom: $\mu\circ\xi_2 = \mu\circ\xi_1 = 0$

$$[1,v] = [v,1] = 0$$

(Bloh 1965, Loday & Cuvier 1991: a non-commutative generalization of Lie algebras.)

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$\rightsquigarrow$ "Leibniz braiding" $\sigma_{Lei}=\tau+\xi\otimes\mu\colon$ $\boxed{v\otimes w\mapsto w\otimes v+1\otimes[v,w]}$

✓ $\boxed{\text{YBE for } \sigma_{Lei} \underset{\text{(Lie un. ax.)}}{\iff} \text{Leibniz identity for } \mu}$

✓ A fully faithful functor $\boxed{\mathbf{Lei}\hookrightarrow\mathbf{Br}_\bullet}$.

4 Alg. structures via braidings

B Leibniz algebras

$(V, \mu: V \otimes V \to V, \xi: \Bbbk \to V)$, $\xi(\alpha) = \alpha 1_V$, s.t.

Leibniz identity: $\mu \circ \mu_2 = \mu \circ \mu_1 - \mu \circ \mu_1 \circ \tau$, where $\tau: w \otimes u \to u \otimes w$

$$[v,[w,u]] = [[v,w],u] - [[v,u],w]$$

Lie unit axiom: $\mu \circ \xi_2 = \mu \circ \xi_1 = 0$

$$[1,v] = [v,1] = 0$$

(Bloh 1965, Loday & Cuvier 1991: a non-commutative generalization of Lie algebras.)

$\rightsquigarrow$ "Leibniz braiding" $\sigma_{Lei} = \tau + \xi \otimes \mu$: $\boxed{v \otimes w \mapsto w \otimes v + 1 \otimes [v,w]}$

✓ $\boxed{\text{YBE for } \sigma_{Lei} \underset{\text{(Lie un. ax.)}}{\iff} \text{Leibniz identity for } \mu}$

✓ A fully faithful functor $\boxed{\mathbf{Lei} \hookrightarrow \mathbf{Br}_\bullet}$.

✓ σ_{Lei} is invertible.

✓ Braided mod. for $(V, \sigma_{Lei}) \longleftrightarrow$ Leibniz mod. for (V, μ, ξ).

4 Alg. structures via braidings

B Leibniz algebras

$(V, \mu\colon V\otimes V \to V, \xi\colon \Bbbk \to V)$, $\xi(\alpha) = \alpha 1_V$, s.t.
$[v,[w,u]] = [[v,w],u] - [[v,u],w]$, $\qquad [1,v] = [v,1] = 0.$

$\rightsquigarrow$ "Leibniz braiding" $\sigma_{\mathrm{Lei}}\colon \boxed{v\otimes w \mapsto w\otimes v + 1\otimes [v,w]}$

✓ Suppose that $V = V' \oplus \Bbbk 1$, $[V',V'] \subseteq V'$. Then
$\Delta_{\mathrm{Lei}} = \xi_1 + \xi_2\colon \boxed{v \mapsto 1\otimes v + v\otimes 1}$ for $v \in V'$ and $\Delta_{\mathrm{Lei}}(1) = 1\otimes 1$
turns $(V, \sigma_{\mathrm{Lei}})$ into a braided coalgebra.

4 Alg. structures via braidings

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✓ Braided homologies for (V,σ_{Lei}) include Leibniz homology.

$$\begin{array}{lccl} & & (M\otimes T(V), d_{Lei}) & \text{Cuvier-Loday} \\ & & \Big\downarrow{\scriptstyle \text{anti-symm.}} & \\ \texttt{Lie} & V \longmapsto & (M\otimes\Lambda(V), d_{CE}) & \text{Chevalley-Eilenberg} \end{array}$$

4 Alg. structures via braidings

Ⓑ Leibniz algebras

$(V,\mu\colon V\otimes V\to V,\xi\colon \Bbbk\to V)$, $\xi(\alpha)=\alpha 1_V$, s.t.
$[v,[w,u]]=[[v,w],u]-[[v,u],w]$, $\qquad [1,v]=[v,1]=0.$

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$$\begin{tikzcd}
\mathbf{Lei} \arrow[d, shift left, twoheadrightarrow, "\text{anti-symm.}"] & V \arrow[r, mapsto] & (M\otimes T(V),d_{Lei}) \arrow[d, twoheadrightarrow, "\text{anti-symm.}"] & \text{Cuvier-Loday} \\
\mathbf{Lie} \arrow[u, shift left, hookrightarrow] & V \arrow[u] \arrow[r, mapsto] & (M\otimes\Lambda(V),d_{CE}) & \text{Chevalley-Eilenberg}
\end{tikzcd}$$

4 Alg. structures via braidings

B Leibniz algebras

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$$\begin{tikzcd}
\mathbf{Lei} \arrow[d, shift left, two heads, "\text{anti-symm.}"] & V \arrow[r, mapsto] & (M\otimes T(V), d_{\mathtt{Lei}}) \arrow[d, two heads, "\text{anti-symm.}"] & \text{Cuvier-Loday} \\
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\end{tikzcd}$$

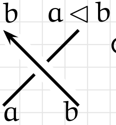
✓ Explains the choice of the lift of the Jacobi identity.

4 Alg. structures via braidings

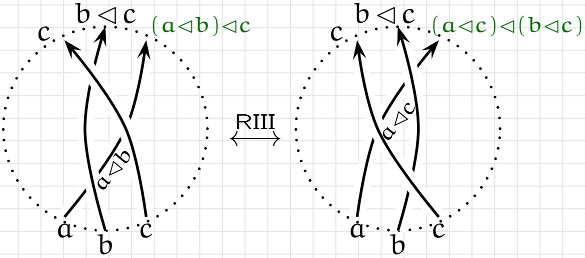
C Self-distributive structures

4 Alg. structures via braidings

C Self-distributive structures



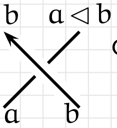
colorings by $(S, \triangleleft)$



RIII $\longleftrightarrow$ $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$ (SD)

4 Alg. structures via braidings

C Self-distributive structures



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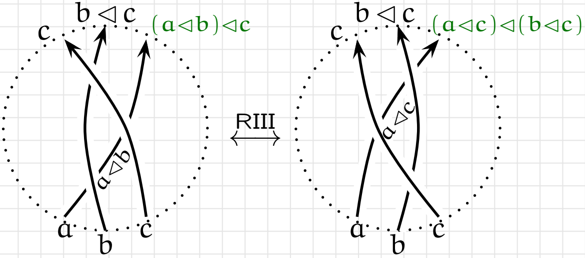


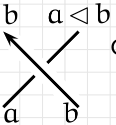
"SD braiding"

$\sigma_{SD}: \boxed{(a,b) \mapsto (b, a \triangleleft b)}$

YBE $\longleftrightarrow$ RIII $\longleftrightarrow$ $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$ (SD)

4 Alg. structures via braidings

C Self-distributive structures



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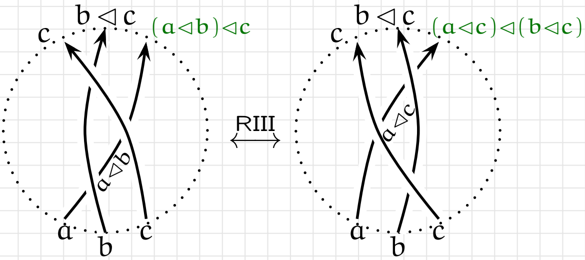
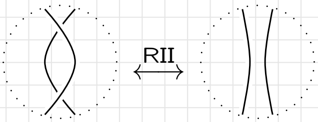
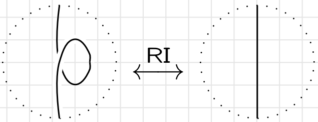


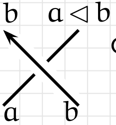
YBE	$\longleftrightarrow$	RIII	$\longleftrightarrow$	$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$	(SD)
$\exists \sigma_{SD}^{-1}$	$\longleftrightarrow$	RII	$\longleftrightarrow$	$a \mapsto a \triangleleft b$ is bijective	(Inv)
		RI	$\longleftrightarrow$	$a \triangleleft a = a$	(Idem)



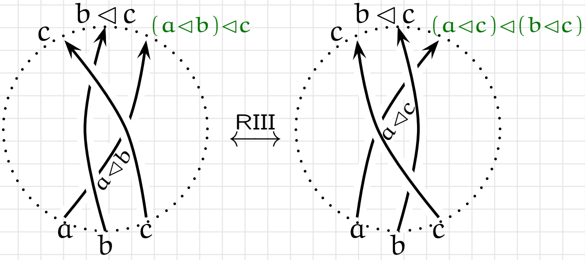


4 Alg. structures via braidings

C Self-distributive structures



colorings by $(S, \lhd)$



"SD braiding"

$\sigma_{SD}: \boxed{(a,b) \mapsto (b, a \lhd b)}$

YBE	$\longleftrightarrow$	RIII	$\longleftrightarrow$	$(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$		(SD)
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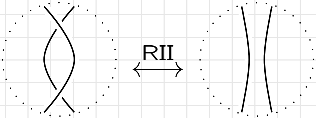
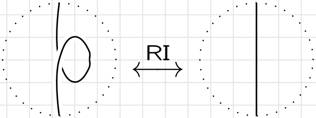
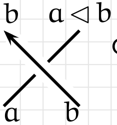




$\Delta_{SD}: \boxed{a \mapsto (a,a)}$

4 Alg. structures via braidings

Ⓒ Self-distributive structures

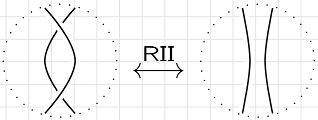


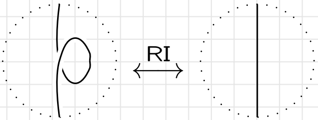
colorings by $(S, \triangleleft)$

Joyce, Matveev 1982:

knot invariants $\overset{\text{colorings}}{\leftsquigarrow}$ quandle

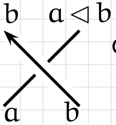
pos. braids	RIII	$\longleftrightarrow$	$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$	shelf
braids	RII	$\longleftrightarrow$	$a \mapsto a \triangleleft b$ is bijective	rack
knots	RI	$\longleftrightarrow$	$a \triangleleft a = a$	quandle





4 Alg. structures via braidings

Ⓒ Self-distributive structures



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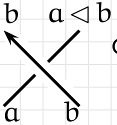
Ex.: → Conjugation quandles: (group G, $g \triangleleft h = h^{-1}gh$)

coloring rule $\longleftrightarrow$ Wirtinger presentation rule,

colorings $\longleftrightarrow$ $\mathrm{Rep}(\pi_1(\mathbb{R}^3 \backslash K), G)$.

4 Alg. structures via braidings

C Self-distributive structures



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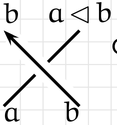
4 Alg. structures via braidings

C Self-distributive structures



colorings by $(S, \lhd)$

Joyce, Matveev 1982:

$$\boxed{\text{knot invariants} \overset{\text{colorings}}{\rightsquigarrow} \text{quandle}}$$

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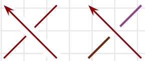
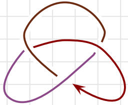
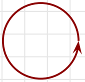
$n = 3$   $\neq$ 

C Self-distributive structures

$$\begin{array}{rccc}
\text{diagrams:} & D & \overset{\text{R-move}}{\rightsquigarrow} & D' \\
\text{colorings:} & \mathcal{C} & \rightsquigarrow & \mathcal{C}' \\
\text{coloring sets:} & \mathcal{C}ol_S(D) & \overset{\text{bij.}}{\longleftrightarrow} & \mathcal{C}ol_S(D') \\
\text{counting invariants:} & \#\mathcal{C}ol_S(D) & = & \#\mathcal{C}ol_S(D')
\end{array}$$

4 Alg. structures via braidings

(C) Self-distributive structures

diagrams:	D	$\overset{\text{R-move}}{\rightsquigarrow}$	D'
colorings:	$\mathcal{C}$	$\rightsquigarrow$	$\mathcal{C}'$
coloring sets:	$\mathcal{Col}_S(D)$	$\overset{\text{bij.}}{\longleftrightarrow}$	$\mathcal{Col}_S(D')$
counting invariants:	$\#\mathcal{Col}_S(D)$	$=$	$\#\mathcal{Col}_S(D')$

Question: Extract more information?

Idea: Some "weight" ω s.t. $\omega(\mathcal{C}) = \omega(\mathcal{C}')$

$$\Longrightarrow \quad \{\omega(\mathcal{C}) \mid \mathcal{C} \in \mathcal{Col}_S(D)\} = \{\omega(\mathcal{C}') \mid \mathcal{C}' \in \mathcal{Col}_S(D')\}.$$

4 Alg. structures via braidings

Ⓒ Self-distributive structures

$$
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$$\Longrightarrow \quad \{\omega(\mathcal{C}) \,|\, \mathcal{C} \in \mathcal{Col}_S(D)\} = \{\omega(\mathcal{C}') \,|\, \mathcal{C}' \in \mathcal{Col}_S(D')\}.$$

Answer: quandle cocycle invariants (Carter-Jelsovsky-Kamada-Langford-Saito 1999).

$\phi : S \times S \to A \ \rightsquigarrow$
Boltzmann weight:

$$\boxed{\omega_\phi(\mathcal{C}) = \sum_{a,\,b} \pm \phi(a, b)}$$

C Self-distributive structures

Rack & quandle cohomology theories

(Fenn-Rourke-Sanderson 1995, Carter et al. 1999)

Motivation:

→ $\{\omega_\phi(\mathcal{C}) \mid \mathcal{C} \in \mathcal{C}ol_S(D)\}$ yields a braid / knot invariant when ϕ is a rack / quandle 2-cocycle;

→ this invariant is trivial when ϕ is a coboundary;

4 Alg. structures via braidings

C Self-distributive structures

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→ adding coefficients allows to color diagram regions:

<u>rack</u> <u>mod.</u> $M \ni \boxed{m} \overset{a}{\uparrow} \boxed{m \cdot a}$

$$\omega_\phi(\mathcal{C}) = \sum \pm\phi(m, a, b)$$

→ everything generalizes to $K^{n-1} \hookrightarrow \mathbb{R}^{n+1}$.

4 Alg. structures via braidings

Ⓒ Self-distributive structures

Rack & quandle cohomology theories

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rack mod. $M \ni \boxed{m} \overset{a}{\uparrow} \boxed{m \cdot a}$

$$\omega_\phi(\mathcal{C}) = \sum_{a,\, m,\, b} \pm\phi(m, a, b)$$

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Inspiration: (Co-)homology theories for associativity.

Question (Przytycki): Explain the parallels between the associative and the SD worlds?

C Self-distributive structures

Rack & quandle cohomology theories

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rack mod. $\quad M \ni \boxed{m} \uparrow^{a} \boxed{m \cdot a}$

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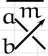

→ everything generalizes to $K^{n-1} \hookrightarrow \mathbb{R}^{n+1}$.

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Question (Przytycki): Explain the parallels between the associative and the SD worlds?

Answer: Common braided interpretation.

4 Alg. structures via braidings

Ⓒ Self-distributive structures

Shelf $(S, \lhd) \rightsquigarrow \sigma_{SD}: \boxed{(a,b) \mapsto (b, a \lhd b)}$

✓ $\boxed{\text{YBE for } \sigma_{SD} \iff \text{SD for } \lhd}$

✓ A fully faithful functor $\boxed{\mathbf{Shelf} \hookrightarrow \mathbf{Br}}$.

✓ σ_{SD} is invertible $\iff (S, \lhd)$ is a rack.

✓ Braided modules for $(V, \sigma_{SD}) \longleftrightarrow$ rack modules for $(S, \lhd)$.

✓ $\Delta_{SD}: \boxed{a \mapsto (a,a)} \rightsquigarrow$ weak braided coalgebra if $(S, \lhd)$ is a quandle.

✓ Braided homologies for (V, σ_{SD}) include rack, quandle, and other SD homologies.

5 Multi-component braidings

Question: How to treat more complicated structures?

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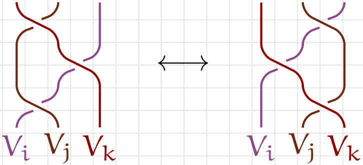
Braided system: $V_1, V_2, \ldots, V_r$ and $\sigma^{i,j} : V_i \otimes V_j \to V_j \otimes V_i,\ i \leqslant j$, satisfying the colored Yang-Baxter equation (cYBE):

$$\boxed{\sigma_1^{j,k} \circ \sigma_2^{i,k} \circ \sigma_1^{i,j} = \sigma_2^{i,j} \circ \sigma_1^{i,k} \circ \sigma_2^{j,k}}$$

$$V_i \otimes V_j \otimes V_k \to V_k \otimes V_j \otimes V_i,\ i \leqslant j \leqslant k$$



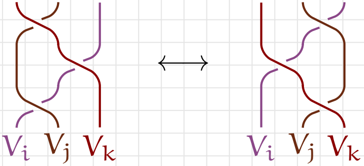
The collection (σ_i) satisfying cYBE is a multi-braiding.

5 Multi-component braidings

Question: How to treat more complicated structures?

Braided system: $V_1, V_2, \ldots, V_r$ and $\sigma^{i,j} : V_i \otimes V_j \to V_j \otimes V_i,\ i \leqslant j$, satisfying the colored Yang-Baxter equation (cYBE):

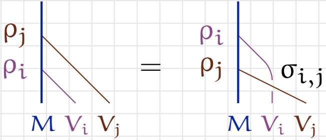
$$\boxed{\sigma_1^{j,k} \circ \sigma_2^{i,k} \circ \sigma_1^{i,j} = \sigma_2^{i,j} \circ \sigma_1^{i,k} \circ \sigma_2^{j,k}}$$



$V_i \otimes V_j \otimes V_k \to V_k \otimes V_j \otimes V_i,\ i \leqslant j \leqslant k$

The collection (σ_i) satisfying cYBE is a multi-braiding.

Left braided V-module:
$(M, (\rho_i : M \otimes V_i \to M))$ s.t.



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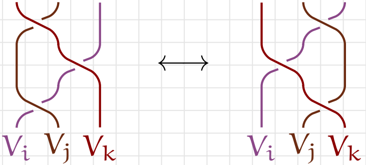

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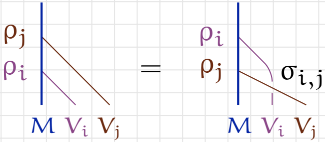

Theorem (L., 2013): $M \otimes T(V_1) \otimes \cdots \otimes T(V_r) \otimes N$ carries a family of differentials $\delta^{(\alpha,\beta)} = \alpha\,{}^{\bullet}\delta + \beta\,\delta^{\bullet}$, $\alpha, \beta \in \mathbb{k}$.

5 Multi-component braidings

Finite-dim. bialgebra $H \rightsquigarrow$

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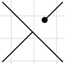
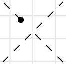
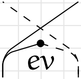
$\sigma_{H,H} =$  $\sigma_{H^*,H^*} =$  $\sigma_{H,H^*} =$ 

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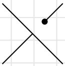
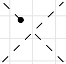
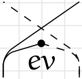
$\sigma_{H,H} =$  $\sigma_{H^*,H^*} =$  $\sigma_{H,H^*} =$ 

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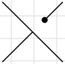
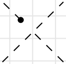
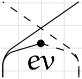
$\sigma_{H,H} =$  $\sigma_{H^*,H^*} =$  $\sigma_{H,H^*} =$ 

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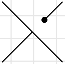
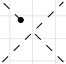
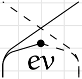
$\sigma_{H,H} =$  $\sigma_{H^*,H^*} =$  $\sigma_{H,H^*} =$ 

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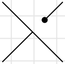
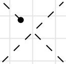
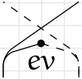
$\sigma_{H,H} =$  $\sigma_{H^*,H^*} =$  $\sigma_{H,H^*} =$ 

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→ Gerstenhaber-Schack; → Panaite-Ştefan.

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✓ Braided homologies include the Ospel-Taillefer theory.

6 The unifying role of the YBE

(multi-)braiding	$\hookleftarrow$	algebraic structure
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(c)YBE	$\Leftrightarrow$	the defining relations
invertibility	$\Leftrightarrow$	algebraic properties
braided modules	$\leftrightarrow$	usual modules
braided hom. theory	$\supseteq$	usual hom. theories

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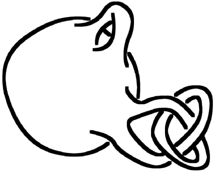
knotted handle-bodies



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