INVERSE SEMI-BRACES AND THE YANG-BAXTER EQUATION

Marzia Mazzotta - Università del Salento (Italy)

Algebra Days in Caen 2022: from Yang–Baxter to Garside, March 24-25 2022

Introduction

In the '90s, Drinfel'd [3] raised the still open issue of finding and classifying all set-theoretical solutions of the Yang-Baxter equation, a fundamental equation of statistical mechanics. If S is a set, a map $r: S \times S \to S \times S$ is said to be a *set theoretical solution of the Yang-Baxter equation*, shortly a *solution*, if the relation

$$\left(r \times \mathrm{id}_S\right)\left(\mathrm{id}_S \times r\right)\left(r \times \mathrm{id}_S\right) = \left(\mathrm{id}_S \times r\right)\left(r \times \mathrm{id}_S\right)\left(\mathrm{id}_S \times r\right)$$

is satisfied. The algebraic structure of *brace* introduced by Rump [5] in 2007 as generalization of Jacobson radical ring is a tool to find solutions. In particular, any brace gives rise to an involutive solution r, i.e., $r^2 = \mathrm{id}$. In this way, Rump traced a novel research direction and later fruitful results on this kind of solutions appeared.
Recently, we introduced the *inverse semi-braces* [2], more general structures than braces, that allow obtaining solutions that are not necessarily bijective.

Inverse semi-braces

We recall that a semigroup S is said to be an *inverse semigroup* if, for each $x \in S$, there exists a unique $x^{-1} \in S$ satisfying $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. In particular, xx^{-1} and $x^{-1}x$ are idempotents.

Definition. Let S be a set endowed of two operations $+$ and $\cdot$ such that $(S, +)$ is a semigroup (not necessarily commutative) and $(S, \cdot)$ is an inverse semigroup. Then, $(S, +, \cdot)$ is an *inverse semi-brace* if

$$a\left(b + c\right) = ab + a\left(a^{-1} + c\right) \tag{1}$$

holds, for all $a, b, c \in S$.

Semi-braces [1, 4] are instances of inverse semi-braces with $(S, \cdot)$ a group. In addition, if $(S, +)$ is an abelian group, then the semi-brace S is a brace.

Example 1. Every inverse semigroup $(S, \cdot)$ having central idempotents gives rise to two inverse semi-braces, setting $a + b = ab$ or $a + b = ba$, for all $a, b \in S$.

Example 2. If $(S, \cdot)$ is an inverse semigroup and $(S, +)$ is a right zero semigroup (or a left zero semigroup), then S is an inverse semi-brace.

Contact Information

marzia.mazzotta@unisalento.it

Solutions associated to inverse semi-braces

Let S be an inverse semi-brace, $\lambda: S \to \mathrm{End}(S, +)$, $a \mapsto \lambda_a$ and $\rho: S \to S^S$, $b \mapsto \rho_b$ the maps respectively defined by

$$\lambda_a\left(b\right) = a\left(a^{-1} + b\right) \qquad \rho_b\left(a\right) = \left(a^{-1} + b\right)^{-1} b,$$

for all $a, b \in S$. Then, we call the map $r_S: S \times S \to S \times S$ given by

$$r_S\left(a, b\right) = \left(\lambda_a\left(b\right), \rho_b\left(a\right)\right),$$

for all $a, b \in S$, the *map associated to the inverse semi-brace* S.

The following are sufficient conditions to obtain solutions through inverse semi-braces.

Theorem 1 Let S be an inverse semi-brace and r_S the map associated to S. If the following conditions are satisfied

1. $\left(a + b\right)\left(a + b\right)^{-1}\left(a + bc\right) = a + bc$
2. $\lambda_a(b)^{-1} + \lambda_{\rho_b(a)}\left(c\right) = \lambda_a(b)^{-1} + \lambda_{\left(a^{-1} + b\right)^{-1}}\lambda_b\left(c\right)$
3. $\rho_b\left(a\right)^{-1} + c = \left(b^{-1} + c\right)\left(\rho_{\lambda_b(c)}\left(a\right)^{-1} + \rho_c\left(b\right)\right)$,

for all $a, b, c \in S$, then the map r_S is a solution.

In general, solutions associated to inverse semi-braces are not bijective.

The previous examples of inverse semi-braces satisfy the conditions of Theorem 1.

Examples

1 The map r_S associated to S in Example 2 with $a + b = ab$ and the map t_S associated to S with $a + b = ba$ are respectively given by

$$r_S(a, b) = \left(aa^{-1}b, b^{-1}ab\right) \qquad t_S(a, b) = \left(aba^{-1}, abb^{-1}\right),$$

and are solutions.

2 The map r_S associated to S in Example 1 with $(S, +)$ a right zero semigroup is given by

$$r_S\left(a, b\right) = \left(ab, b^{-1}b\right)$$

and is an idempotent solution. Similarly, if $(S, +)$ a left zero semigroup, we get the idempotent solution

$$r_S\left(a, b\right) = \left(aa^{-1}, ab\right).$$

Note that if $|S| > 1$ such solutions are not isomorphic. In addition, it is clear that the number of inverse semigroups determines a lower bound for idempotent solutions.

The double semidirect product of inverse semi-braces

Theorem 2 Let S and T be two inverse semi-braces, $\sigma: T \to \mathrm{Aut}\left(S\right)$ a homomorphism from $(T, \cdot)$ into the automorphism group of the inverse semi-brace S, and $\delta: S \to \mathrm{End}\left(T\right)$ an anti-homomorphism from $(S, +)$ into the endomorphism semigroup of $(T, +)$. Set ${}^{u}a := \sigma(u)(a)$ and $u^a := \delta(a)(u)$, for all $a \in S$ and $u \in T$, if it holds

$$\left(uv\right)^{\lambda_a\left({}^{u}b\right)} + u\left(\left(u^{-1}\right)^b + w\right) = u\left(v^b + w\right), \tag{2}$$

then $B := S \times T$ with respect to the operations

$$\left(a, u\right) + \left(b, v\right) := \left(a + b,\ u^b + v\right) \qquad \left(a, u\right)\left(b, v\right) := \left(a\,{}^{u}b,\ uv\right),$$

is an inverse semi-brace. We call such an inverse semi-brace B the *double semidirect product* of S and T via σ and δ.

Set $\Omega^a_{u,v} := \left(u^{-1}\right)^a + v$, for all $a \in S$, $u, v \in T$, the map r_B associated to B is given by

$$r_B\left(\left(a, u\right), \left(b, v\right)\right) = \left(\left(\lambda_a\left({}^{u}b\right), u\,\Omega^b_{u,v}\right), \left({}^{\left(\Omega^b_{u,v}\right)^{-1}u^{-1}}\rho_{{}^{u}b}\left(a\right),\ \left(\Omega^b_{u,v}\right)^{-1}v\right)\right).$$

In the particular case of S and T semi-braces, we have the following result.

Theorem 3 Let S, T be semi-braces and B the double semidirect product of S and T via σ and δ. If r_S and r_T are solutions associated to S and T, respectively, and the following are satisfied

1. $\left(u^1\right)^a = u^a$,
2. $1^a + u = 1 + u$,

for all $a \in S$ and $u \in T$, then the map r_B associated to B is a solution.

References

[1] F. Catino, I. Colazzo, and P. Stefanelli. Semi-braces and the Yang-Baxter equation. *J. Algebra*, 483:163–187, 2017.

[2] F. Catino, M. Mazzotta, and P. Stefanelli. Inverse semi-braces and the Yang-Baxter equation. *J. Algebra*, 573:576–619, 2021.

[3] V. G. Drinfel′ d. On some unsolved problems in quantum group theory. In *Quantum groups (Leningrad, 1990)*, volume 1510 of *Lecture Notes in Math.*, pages 1–8. Springer, Berlin, 1992.

[4] E. Jespers and A. Van Antwerpen. Left semi-braces and solutions of the Yang-Baxter equation. *Forum Math.*, 31(1):241–263, 2019.

[5] W. Rump. Braces, radical rings, and the quantum Yang-Baxter equation. *J. Algebra*, 307(1):153–170, 2007.