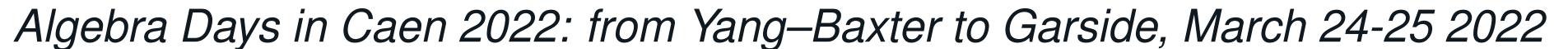
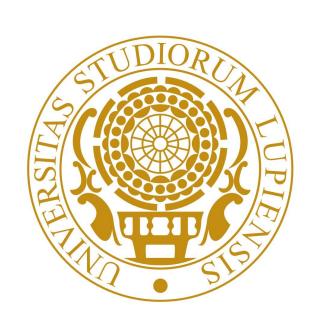
INVERSE SEMI-BRACES AND THE YANG-BAXTER EQUATION

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Introduction

In the '90s, Drinfel'd [3] raised the still open issue of finding and classifying all set-theoretical solutions of the Yang-Baxter equation, a fundamental equation of statistical mechanics. If S is a set, a map $r: S \times S \to S \times S$ is said to be a *set theoretical solution of the Yang-Baxter equation*, shortly a *solution*, if the relation

$$(r \times id_S) (id_S \times r) (r \times id_S) = (id_S \times r) (r \times id_S) (id_S \times r)$$

is satisfied. The algebraic structure of *brace* introduced by Rump [5] in 2007 as generalization of Jacobson radical ring is a tool to find solutions. In particular, any brace gives rise to an involutive solution r, i.e., $r^2 = \mathrm{id}$. In this way, Rump traced a novel research direction and later fruitful results on this kind of solutions appeared.

Recently, we introduced the *inverse semi-braces* [2], more general structures than braces, that allow obtaining solutions that are not necessarily bijective.

Inverse semi-braces

We recall that a semigroup S is said to be an *inverse semigroup* if, for each $x \in S$, there exists a unique $x^{-1} \in S$ satisfying $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. In particular, xx^{-1} and $x^{-1}x$ are idempotents.

Definition. Let S be a set endowed of two operations + and \cdot such that (S,+) is a semigroup (not necessarily commutative) and (S,\cdot) is an inverse semigroup. Then, $(S,+,\cdot)$ is an *inverse semi-brace* if

$$a(b+c) = ab + a(a^{-1} + c)$$
 (1)

holds, for all $a, b, c \in S$.

Semi-braces [1, 4] are instances of inverse semi-braces with (S, \cdot) a group. In addition, if (S, +) is an abelian group, then the semi-brace S is a brace.

Example 1. Every inverse semigroup (S, \cdot) having central idempotents gives rise to two inverse semi-braces, setting a+b=ab or a+b=ba, for all $a,b\in S$.

Example 2. If (S, \cdot) is an inverse semigroup and (S, +) is a right zero semigroup (or a left zero semigroup), then S is an inverse semi-brace.

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Solutions associated to inverse semi-braces

Let S be an inverse semi-brace, $\lambda: S \to \operatorname{End}(S,+), a \mapsto \lambda_a$ and $\rho: S \to S^S, b \mapsto \rho_b$ the maps respectively defined by

$$\lambda_a(b) = a(a^{-1} + b)$$
 $\rho_b(a) = (a^{-1} + b)^{-1}b,$

for all $a, b \in S$. Then, we call the map $r_S : S \times S \to S \times S$ given by

$$r_S(a,b) = (\lambda_a(b), \rho_b(a)),$$

for all $a, b \in S$, the map associated to the inverse semi-brace S.

The following are sufficient conditions to obtain solutions through inverse semi-braces.

Theorem 1 Let S be an inverse semi-brace and r_S the map associated to S. If the following conditions are satisfied

1.
$$(a+b)(a+b)^{-1}(a+bc) = a+bc$$

2.
$$\lambda_a(b)^{-1} + \lambda_{\rho_b(a)}(c) = \lambda_a(b)^{-1} + \lambda_{(a^{-1}+b)^{-1}}\lambda_b(c)$$

3.
$$\rho_b(a)^{-1} + c = (b^{-1} + c) \left(\rho_{\lambda_b(c)}(a)^{-1} + \rho_c(b) \right)$$
,

for all $a, b, c \in S$, then the map r_S is a solution.

In general, solutions associated to inverse semi-braces are not bijective.

The previous examples of inverse semi-braces satisfy the conditions of Theorem 1.

Examples

1 The map r_S associated to S in Example 2 with a+b=ab and the map t_S associated to S with a+b=ba are respectively given by

$$r_S(a,b) = (aa^{-1}b, b^{-1}ab)$$
 $t_S(a,b) = (aba^{-1}, abb^{-1}),$

and are solutions.

2 The map r_S associated to S in Example 1 with (S,+) a right zero semigroup is given by

$$r_S(a,b) = (ab, b^{-1}b)$$

and is an idempotent solution. Similarly, if (S,+) a left zero semigroup, we get the idempotent solution

$$r_S(a,b) = (aa^{-1},ab)$$
.

Note that if |S| > 1 such solutions are not isomorphic. In addition, it is clear that the number of inverse semigroups determines a lower bound for idempotent solutions.

The double semidirect product of inverse semi-braces

Theorem 2 Let S and T be two inverse semi-braces, $\sigma: T \to \operatorname{Aut}(S)$ a homomorphism from (T,\cdot) into the automorphism group of the inverse semi-brace S, and $\delta: S \to \operatorname{End}(T)$ an anti-homomorphism from (S,+) into the endomorphism semigroup of (T,+). Set $^ua:=\sigma(u)(a)$ and $u^a:=\delta(a)(u)$, for all $a\in S$ and $u\in T$, if it holds

$$(uv)^{\lambda_a(ub)} + u\left((u^{-1})^b + w\right) = u\left(v^b + w\right), \tag{2}$$

then $B := S \times T$ with respect to the operations

$$(a, u) + (b, v) := (a + b, u^b + v)$$
 $(a, u) (b, v) := (a^u b, uv),$

is an inverse semi-brace. We call such an inverse semi-brace B the double semidirect product of S and T via σ and δ .

Set $\Omega_{u,v}^a:=\left(u^{-1}\right)^a+v$, for all $a\in S,\,u,v\in T$, the map r_B associated to B is given by

$$r_{B}\left(\left(a,u\right),\left(b,v\right)\right)=\left(\left(\lambda_{a}\left({}^{u}b\right),u\,\Omega_{u,v}^{b}\right),\left(\left(\Omega_{u,v}^{b}\right)^{-1}u^{-1}\rho_{u_{b}}\left(a\right),\left(\Omega_{u,v}^{b}\right)^{-1}v\right)\right).$$

In the particular case of S and T semi-braces, we have the following result.

Theorem 3 Let S, T be semi-braces and B the double semidirect product of S and T via σ and δ . If r_S and r_T are solutions associated to S and T, respectively, and the following are satisfied

1.
$$(u^1)^a = u^a$$
,

2.
$$1^a + u = 1 + u$$
,

for all $a \in S$ and $u \in T$, then the map r_B associated to B is a solution.

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