Unexpected applications of homotopical algebra to knot theory

(Honest title: Homology of permutation racks)

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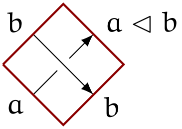
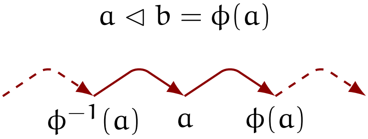


I. From topology to algebra



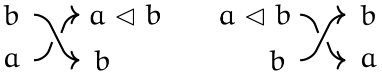
II. From algebra to topology

1 How topologists discovered self-distributivity

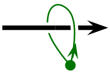
D. Joyce & S. Matveev, knot colorists separated by the Iron Curtain:

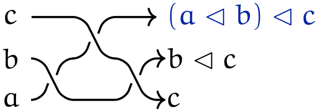
Take a set S endowed with a binary operation $\triangleleft$.

$(S, \triangleleft)$-colourings for braid diagrams:

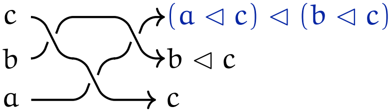


cf. Wirtinger presentation of $\pi_1([0,1] \times \mathbb{R}^2 \setminus \beta)$:





RIII
$\sim$



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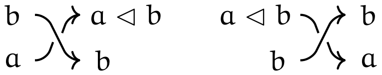
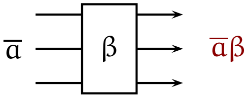


$\mathrm{End}(S^n) \leftarrow B_n^+$	RIII	$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$	shelf
$\mathrm{Aut}(S^n) \leftarrow B_n$	& RII	$\forall b,\ a \mapsto a \triangleleft b$ is bijective	rack
$S \hookrightarrow (S^n)^{B_n}$	& RI	$a \triangleleft a = a$	quandle

$a \mapsto (a, \ldots, a)$



2 Braids and self-distributivity

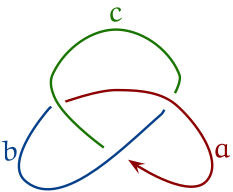
$\mathrm{End}(S^n) \leftarrow B_n^+$	RIII	$(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$	shelf
$\mathrm{Aut}(S^n) \leftarrow B_n$	& RII	$\forall b,\ a \mapsto a \lhd b$ is bijective	rack
$S \hookrightarrow (S^n)^{B_n}$	& RI	$a \lhd a = a$	quandle

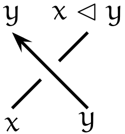
$a \mapsto (a, \ldots, a)$

S	$a \lhd b$	$(S, \lhd)$ is a	in braid theory
${}_{\mathbb{Z}[t^{\pm 1}]}\mathrm{Mod}$	$ta + (1-t)b$	quandle	Burau: $B_n \to GL_n(\mathbb{Z}[t^{\pm}])$
group	$b^{-1}ab$	quandle	Artin: $B_n \hookrightarrow \mathrm{Aut}(F_n)$
twisted linear quandle			Lawrence–Krammer–Bigelow
$\mathbb{Z}$	$a + 1$	rack	$\lg(w), \mathrm{lk}_{i,j}$
free shelf			Dehornoy: order on B_n

3 Knots and self-distributivity

$(S, \triangleleft)$-colourings for knot diagrams:

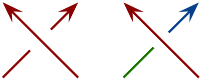


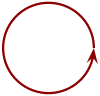


$$a \triangleleft b = c, \quad b \triangleleft c = a, \quad c \triangleleft a = b$$

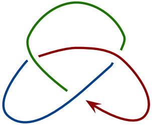
Proposition: $(S, \triangleleft)$ is a quandle $\implies$
$\#\{\,(S, \triangleleft)\text{-colourings of diagrams}\,\}$ is a knot invariant.

Example: $(\mathbb{Z}_3,\ a \triangleleft b = 2b - a)$



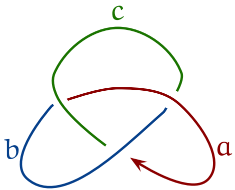


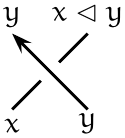
3 colourings



9 colourings

$(S, \triangleleft)$-colourings for knot diagrams:



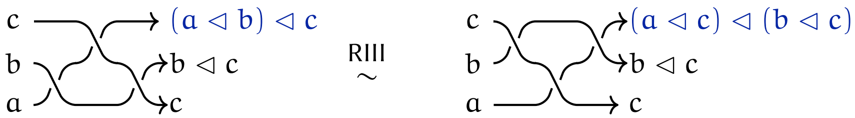


$$a \triangleleft b = c, \quad b \triangleleft c = a, \quad c \triangleleft a = b$$

Theorem (*Joyce & Matveev '82*):

- $\#\,\mathrm{Col}_{S,\triangleleft}(D) = \#\,\mathrm{Hom}_{\mathtt{Quandle}}(Q(K), S)$,
- $Q(K)$ = fundamental quandle of K (a weak universal knot invariant).

4 The homology comes in



diagrams:	D	$\overset{\text{R-move}}{\rightsquigarrow}$	D'
colorings:	$\mathcal{C}$	$\rightsquigarrow$	$\mathcal{C}'$
coloring sets:	$\mathrm{Col}_{S,\triangleleft}(D)$	$\overset{1:1}{\longleftrightarrow}$	$\mathrm{Col}_{S,\triangleleft}(D')$

Counting invariants: $\#\,\mathrm{Col}_{S,\triangleleft}(D) = \#\,\mathrm{Col}_{S,\triangleleft}(D')$.

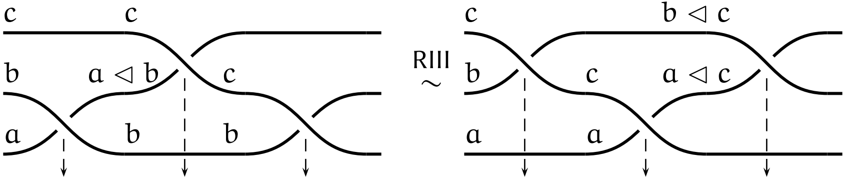
Question: Extract more information?

$$\omega(\mathcal{C}) = \omega(\mathcal{C}')$$
$$\Downarrow$$
$$\{\,\omega(\mathcal{C}) \,|\, \mathcal{C} \in \mathrm{Col}_{S,\triangleleft}(D)\,\} = \{\,\omega(\mathcal{C}') \,|\, \mathcal{C}' \in \mathrm{Col}_{S,\triangleleft}(D')\,\}.$$

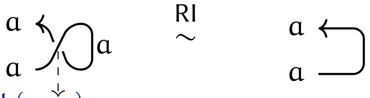
Answer (*Carter–Jelsovsky–Kamada–Langford–Saito* '03): State-sums over crossings, and Boltzmann weights:

$$\phi\colon S \times S \to \mathbb{Z}_m \qquad \rightsquigarrow \qquad \omega_\phi(\mathcal{C}) = \sum \pm\phi(a, b)$$

Conditions on ϕ:



$$\phi(a, b) + \phi(a \lhd b, c) + \cancel{\phi(b, c)} = \cancel{\phi(b, c)} + \phi(a, c) + \phi(a \lhd c, b \lhd c)$$



$$\phi(a, a) = 0$$

Quandle cycle invariants: $\{\, \omega_\phi(\mathcal{C}) \mid \mathcal{C} \in \mathrm{Col}_{S,\lhd}(D) \,\}$.

$$\phi\colon S \times S \to \mathbb{Z}_m \qquad \rightsquigarrow \qquad \omega_\phi(\mathcal{C}) = \sum \pm\phi(a, b)$$

Quandle cocycle invariants: $\left\{ \omega_\phi(\mathcal{C}) \,\middle|\, \mathcal{C} \in \mathrm{Col}_{S,\triangleleft}(D) \right\}$.

Example: $\phi = 0 \qquad \rightsquigarrow \qquad$ counting invariants.

Quandle cocycle invariants $\supsetneq$ counting invariants.

Conjecture (*Clark–Saito–…*):

Finite quandle cocycle invariants distinguish all knots.

Generalisation: $K^n \hookrightarrow \mathbb{R}^{n+2}$ and $\phi\colon S^{\times(n+1)} \to \mathbb{Z}_m$.

Wish:

$d^{n+1}\phi = 0 \implies \phi$ refines counting invariants for n-knots,
$\phi = d^n\psi \implies$ the refinement is trivial.

5 The desired cohomology theory

Fenn et al. '95 & *Carter et al.* '03 & *Graña* '00:

Shelf $(S, \lhd)$ & abelian group $X \rightsquigarrow$ cochain complex

$$C^k_R(S,X) = \mathrm{Map}(S^{\times k}, X),$$

$$(d^k_R f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} (f(a_1, \ldots, \widehat{a_i}, \ldots, a_{k+1}) - f(a_1 \lhd a_i, \ldots, a_{i-1} \lhd a_i, a_{i+1}, \ldots, a_{k+1}))$$

$\rightsquigarrow$ Rack cohomology $H^k_R(S,X) = \mathrm{Ker}\, d^k_R / \mathrm{Im}\, d^{k-1}_R$.

Quandle $(S, \lhd)$ & abelian group $X \rightsquigarrow$ sub-complex of (C^k_R, d^k_R):

$$C^k_Q(S,X) = \{ f \colon S^{\times k} \to X \mid f(\ldots, a, a, \ldots) = 0 \}$$

$\rightsquigarrow$ Quandle cohomology $H^k_Q(S,X)$.

This is what we were looking for! This construction yields:

- ✓ Boltzmann weights for constructing higher knot invariants (powerfull and easy to compute);
- ✓ an important class of braided vector spaces giving nice Hopf algebras;
- ✓ a parametrization of abelian rack extensions.

Problem: Full rack/quandle (co)homology of a rack is hard to compute.

We will give a partial overview of available tools.

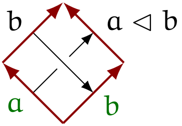
6 Topological realization

Fenn–Rourke–Sanderson '95:

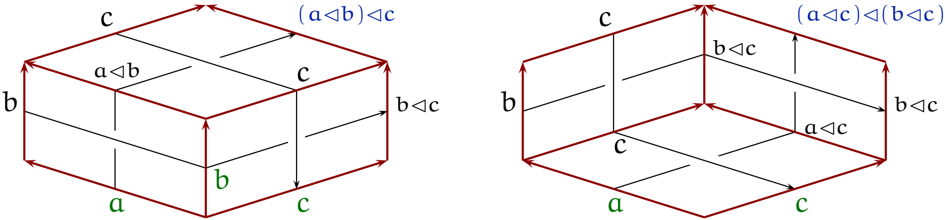
Shelf $(S, \triangleleft)$ $\rightsquigarrow$ rack (= classifying) space $B(S)$. It is a CW-complex:

deg 0: $*$ deg 1: $* \xrightarrow{a} *$

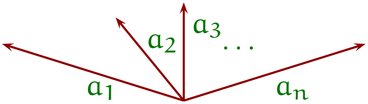
deg 2:



deg 3:



deg n: $\coprod_{S^{\times n}} [0,1]^n$

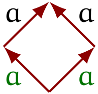


The coloring continues uniquely to other edges of $[0,1]^n$.

Boundaries: usual topological ones.

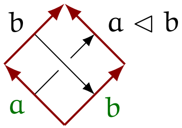
$$\boxed{H^\bullet_R(S, X) \cong H^\bullet(B(S), X)}$$

Nosaka '11: To get quandle cohomology, add 3-dimensional cells bounding



Proposition: $\boxed{\pi_1(B(S)) \cong \mathrm{As}(S)}$,

where $\mathrm{As}(S) := \langle S \mid a\,b \;=\; b\,(a \lhd b)\rangle$ is the associated group of $(S, \lhd)$.



Computations (*Fenn–Rourke–Sanderson* '07):

1) Trivial quandle $T_n = (\{1, \ldots, n\},\; a \lhd b = a)$: $\quad B(T_n) \cong \Omega(\vee_n \mathbb{S}^2)$.

2) Free rack on n generators FR_n: $\quad B(FR_n) \cong \vee_n \mathbb{S}^1$.

7 Rack cohomology vs group cohomology

The associated group of $(S, \triangleleft)$:

$$\mathrm{As}(S) := \langle S \mid a\, b \;=\; b\,(a \triangleleft b)\rangle$$

Theorem (*Joyce* '82): One has a pair of adjoint functors

$$\mathrm{As} : \textbf{Rack} \rightleftarrows \textbf{Group} : \mathrm{Conj}\,.$$

Theorem (*García Iglesias & Vendramin* '16): For a finite indecomposable quandle S,

$$H^2_{\mathrm{R}}(S, X) \cong X \times \mathrm{Hom}(N(S), X).$$

Here $N(S)$ is a finite group (the stabilizer of an $a_0 \in S$ in $[\mathrm{As}(S), \mathrm{As}(S)]$).

Theorem (*Fenn–Rourke–Sanderson* '95): There is a graded algebra morphism $HH^{\bullet}(\mathrm{As}(S), X) \to H^{\bullet}_{\mathrm{R}}(S, X)$.

8 Betti numbers

Theorem (*Etingof–Graña* '03): If $(S, \triangleleft)$ is a rack and $\#\,\mathrm{Inn}(S) \in X^*$, then

$$\boxed{H_R^k(S, X) \cong X^{r^k}}$$

- ✓ $\mathrm{Orb}(S) = \{$ orbits of S w.r.t. $a \sim a \triangleleft b\}$, $r = \#\,\mathrm{Orb}(S)$;
- ✓ $\mathrm{Inn}(S)$ is the subgroup of $\mathrm{Aut}(S)$ generated by $t_b \colon a \mapsto a \triangleleft b$.

Bad news: If $\#\,\mathrm{Inn}(S) \in X^*$, then

quandle cocycle invariants = coloring invariants + linking numbers.

Hope: Look at $X = \mathbb{Z}_p$, or at the p-torsion of $H_R^k(S, \mathbb{Z})$, where $p \mid \#\,\mathrm{Inn}(S)$.

It works, and yields interesting invariants!

9 Homotopical tools: framework

Theorem (*Szymik* '19): Quandle cohomology is a Quillen cohomology.

Applications:

- ✓ excision isomorphisms;
- ✓ Mayer–Vietoris exact sequences.

10 Homotopical tools: example

A permutation ϕ on a set S $\rightsquigarrow$ permutation rack $(S\,,\ a \lhd_\phi b = \phi(a))$.

Theorem (*L.–Szymik* '20): $\boxed{H^R_k((S, \lhd_\phi), X) \cong X^{\beta_k}}$ where

- ✓ $\beta_0 = 1,\ \beta_1 = r,\ \beta_{n+2} = (r-1)\beta_{n+1} + r_f \beta_n, \qquad n \geqslant 0$;
- ✓ $r = \#\{$ orbits of ϕ $\}$, $\qquad r_f = \#\{$ finite orbits of ϕ $\}$.

Remark: $H^R_\bullet(S, \lhd_\phi)$ contains more information than $\mathrm{As}(S, \lhd_\phi)$.

Sketch of proof:

Step 1 Explicit computations for free permutation racks
(= all orbits are infinite).

Trick: $H^k_R = \mathrm{Ker}\, d^k_R / \mathrm{Im}\, d^{k-1}_R$

study chains up to boundaries, then restrict to cycles
(usually: determine cycles, then mod out the boundaries).

Step 2 Choose a simplicial resolution by free permutations $F_\bullet \to S$

$\rightsquigarrow$ a double complex $E^0_{p,q} = C^R_q(F_p)$

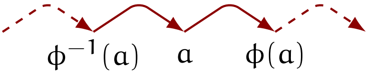
$\rightsquigarrow$ two spectral sequences with the same target.

Step 3 Computations in the spectral sequences:

1st SS: $E^\infty_{p,q} \cong \begin{cases} H^R_q(S) & \text{if } p = 0, \\ 0 & \text{if } p \neq 0. \end{cases}$.

2nd SS: $E^2_{\bullet,q} \cong \overline{H}_\bullet(S /\!\!/ \phi)^{\otimes(q-1)} \otimes H_\bullet(S /\!\!/ \phi)$,

where $S /\!\!/ \phi$ is the homotopy orbit space:



$S /\!\!/ \phi = r_f$ circles $\bigsqcup$ $r - r_f$ lines

Step 4 For the 2nd SS, show that $E^\infty = E^2$.

For this, find enough independent elements in $H^R_q(S)$.