Homotopical tools for computing rack homology

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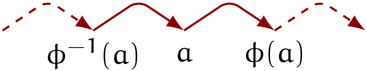
$$(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$$

$$a \lhd b = \phi(a)$$

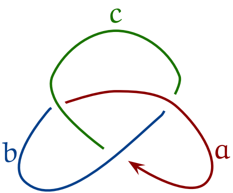


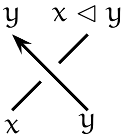
1 How topologists discovered self-distributivity

D. Joyce & S. Matveev, knot colorists separated by the Iron Curtain:

Take a set S endowed with a binary operation $\triangleleft$.

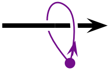
$(S, \triangleleft)$-colourings for knot diagrams:





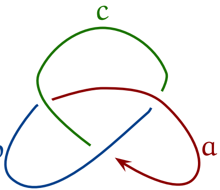
$$a \triangleleft b = c, \quad b \triangleleft c = a, \quad c \triangleleft a = b$$

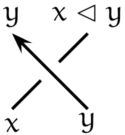
cf. Wirtinger presentation of $\pi_1(\mathbb{R}^3 \setminus K)$:



$$x \triangleleft y = y^{-1}xy$$

$(S, \triangleleft)$-colourings for knot diagrams:





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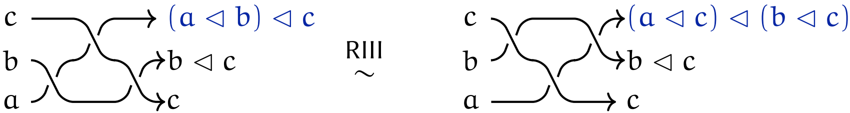
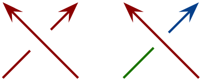


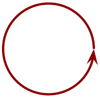
RIII	$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$	shelf
RII	$\forall b,\ a \mapsto a \triangleleft b$ is bijective	rack
RI	$a \triangleleft a = a$	quandle

Proposition: $(S, \triangleleft)$ is a quandle $\Longrightarrow$

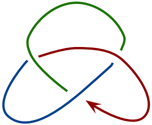
$\# \{ (S, \triangleleft)\text{-colourings of diagrams} \}$ is a knot invariant.

Example: $(\mathbb{Z}_3,\ a \triangleleft b = 2b - a)$





3 colourings



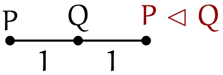
9 colourings

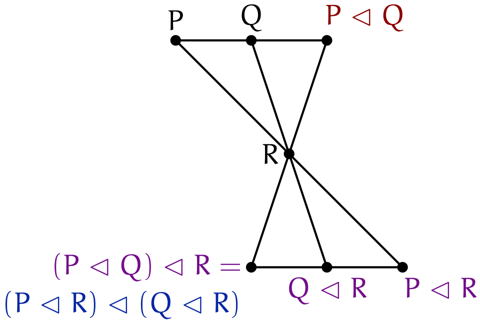
Theorem (*Joyce & Matveev '82*):

These invariants have a good reason to be strong.

2 An example of a quandle

Mituhisa Takasaki, a fresh Japanese maths PhD in 1940 Harbin, and *Gavin Wraith*, a bored US school boy in the '50s:





More generally: geometric symmetries.

Another **example** you might like: Coxeter racks

$(V \setminus \{0\},\ a \lhd b = a - 2\frac{\langle a,b \rangle}{\langle b,b \rangle} b)$,

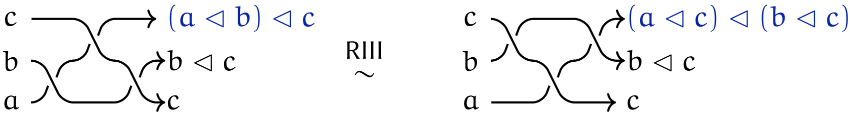
where V is a vector spaces endowed with a nice form.

3 Braids and self-distributivity

One can play the same game with braids, and obtain actions of the braid groups B_n out of rack colourings.

S	$a \lhd b$	$(S, \lhd)$	in braid theory it yields
${}_{\mathbb{Z}[t^{\pm 1}]}\mathrm{Mod}$	$ta + (1-t)b$	Al. quandle	Burau: $B_n \to GL_n(\mathbb{Z}[t^{\pm}])$
group	$b^{-1}ab$	conj. quandle	Artin: $B_n \hookrightarrow \mathrm{Aut}(F_n)$
twisted Alexander quandle			Lawrence–Krammer–Bigelow
$\mathbb{Z}$	$a+1$	rack	$lg(w)$, $lk_{i,j}$
free shelf			Dehornoy: order on B_n

4 The homology comes in



$$\begin{array}{rccc} \text{diagrams:} & D & \overset{\text{R-move}}{\rightsquigarrow} & D' \\ \text{colorings:} & \mathcal{C} & \rightsquigarrow & \mathcal{C}' \\ \text{coloring sets:} & \mathrm{Col}_{S,\triangleleft}(D) & \overset{1:1}{\longleftrightarrow} & \mathrm{Col}_{S,\triangleleft}(D') \end{array}$$

Counting invariants: $\#\,\mathrm{Col}_{S,\triangleleft}(D) = \#\,\mathrm{Col}_{S,\triangleleft}(D')$.

Question: Extract more information?

$$\omega(\mathcal{C}) = \omega(\mathcal{C}')$$

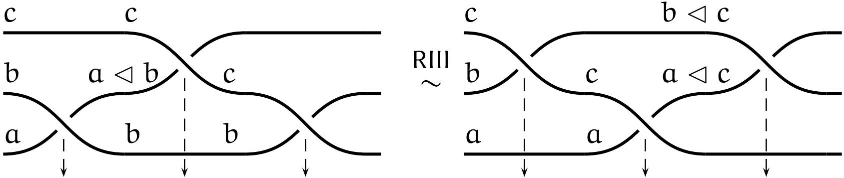
$$\Downarrow$$

$$\{\, \omega(\mathcal{C}) \,|\, \mathcal{C} \in \mathrm{Col}_{S,\triangleleft}(D) \,\} = \{\, \omega(\mathcal{C}') \,|\, \mathcal{C}' \in \mathrm{Col}_{S,\triangleleft}(D') \,\}.$$

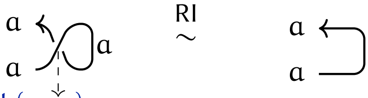
Answer (*Carter–Jelsovsky–Kamada–Langford–Saito* '03): State-sums over crossings, and Boltzmann weights:

$$\phi\colon S \times S \to \mathbb{Z}_m \qquad \rightsquigarrow \qquad \omega_\phi(\mathcal{C}) = \sum \pm\phi(a,b)$$

Conditions on ϕ:



$$\phi(a,b)+\phi(a \triangleleft b, c)+\cancel{\phi(b,c)} = \cancel{\phi(b,c)}+\phi(a,c)+\phi(a \triangleleft c, b \triangleleft c)$$



$$\phi(a,a) = 0$$

Quandle cycle invariants: $\{\, \omega_\phi(\mathcal{C}) \mid \mathcal{C} \in \mathrm{Col}_{S,\triangleleft}(D) \,\}$.

$$\phi\colon S \times S \to \mathbb{Z}_m \qquad \rightsquigarrow \qquad \omega_\phi(\mathcal{C}) = \sum \pm\phi(a, b)$$

Quandle cocycle invariants: $\{\, \omega_\phi(\mathcal{C}) \mid \mathcal{C} \in \mathrm{Col}_{S,\triangleleft}(D) \,\}$.

Example: $\phi = 0 \qquad \rightsquigarrow \qquad$ counting invariants.

Quandle cocycle invariants $\supsetneq$ counting invariants.

Generalisation: $K^n \hookrightarrow \mathbb{R}^{n+2}$ and $\phi\colon S^{\times(n+1)} \to \mathbb{Z}_m$.

Wish:

$d^{n+1}\phi = 0 \implies \phi$ refines counting invariants for n-knots,

$\phi = d^n\psi \implies$ the refinement is trivial.

5 The desired cohomology theory

Fenn et al. '95 & *Carter et al.* '03 & *Graña* '00:

Shelf $(S, \lhd)$ & abelian group X $\rightsquigarrow$ cochain complex

$$C^k_R(S, X) = \mathrm{Map}(S^{\times k}, X),$$

$$(d^k_R f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} (f(a_1, \ldots, \widehat{a_i}, \ldots, a_{k+1})$$

$$- f(a_1 \lhd a_i, \ldots, a_{i-1} \lhd a_i, a_{i+1}, \ldots, a_{k+1}))$$

$\rightsquigarrow$ Rack cohomology $H^k_R(S, X) = \mathrm{Ker}\, d^k_R / \mathrm{Im}\, d^{k-1}_R$.

Quandle $(S, \lhd)$ & abelian group X $\rightsquigarrow$ sub-complex of (C^k_R, d^k_R):

$$C^k_Q(S, X) = \{ f \colon S^{\times k} \to X \mid f(\ldots, a, a, \ldots) = 0 \}$$

$\rightsquigarrow$ Quandle cohomology $H^k_Q(S, X)$.

This is what we were looking for! This construction yields:

- ✓ Boltzmann weights for constructing higher knot invariants (powerful and easy to compute);
- ✓ a parametrisation of abelian rack extensions;
- ✓ an important class of braided vector spaces giving nice Hopf algebras.

6 How Hopf algebraists discovered SD

Very open question: Classify f.-d. pointed Hopf algebras over $\mathbb{C}$.

Applications:

- ✓ cohomology of H-spaces, e.g. Lie groups (*Hopf* '41);
- ✓ invariants of knots and 3-manifolds, TQFT;
- ✓ non-commutative geometry;
- ✓ condensed-matter physics, string theory,

....

Examples:

- ✓ group algebras $\Bbbk G$;
- ✓ enveloping algebras of Lie algebras $U(\mathfrak{g})$;
- ✓ quantum groups: deformations $U_q(\mathfrak{g})$ for semisimple $\mathfrak{g}$,

....

Classification program (*Andruskiewitsch–Graña–Schneider* '98):

$$\begin{array}{ccc}
\text{nice Hopf algebra } A & & \\
\rightsquigarrow & & \\
\text{Yetter–Drinfel'd module } V \in {}^H_H\mathcal{YD} & & \\
\color{red}{\rightsquigarrow} & & \\
\text{braided vector space } (V,\sigma) & \color{red}{\leftsquigarrow} & \text{rack } (S,\triangleleft)\ \&\ \phi\colon S\times S\to\mathbb{Z}_m \\
\rightsquigarrow & & \\
\text{Nichols algebra } \mathcal{B}(V) & & \\
\rightsquigarrow & & \&\ V\in {}^H_H\mathcal{YD} \\
\text{bosonization Hopf algebra } \mathcal{B}(V)\#H & &
\end{array}$$

✓ $G(A) =$ the group of group-like elements of A, $H(A) = \mathbb{C}G(A)$;

✓ $R(A) =$ coinvariants of $\mathrm{gr}(A) \twoheadrightarrow \mathrm{gr}(A)_0 = H(A)$, $V(A) = \mathrm{Prim}(R(A))$;

✓ $\sigma \in \mathrm{Aut}(V \otimes V)$, $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$,

where $\sigma_1 = \sigma \times \mathrm{Id}_S$, $\sigma_2 = \mathrm{Id}_S \times \sigma$;

✓ in red: "arrows with a large image";

✓ $\mathrm{gr}(A) \cong R(A)\#H(A) =$ [conjecturally] $= \mathcal{B}(V(A))\#H(A)$.

braided vector space $(\mathbb{C}S, \sigma_{\lhd,\phi})$ $\leftsquigarrow$ rack $(S, \lhd)$ & $\phi \colon S \times S \to \mathbb{Z}_m$

$$\boxed{\sigma_{\lhd,\phi} \colon (a, b) \mapsto q^{\phi(a,b)}(b, a \lhd b)}$$

Here q is an mth root of unity, or transcendental.

Wish:

$d^2\phi = 0 \implies (\mathbb{C}S, \sigma_{\lhd,\phi})$ is a braided vector space,

$\phi - \phi' = d^1\psi \implies$ the braided vector spaces are isomorphic.

7 Topological realization

Fenn–Rourke–Sanderson '95:

Shelf $(S, \lhd)$ $\rightsquigarrow$ classifying space $B(S)$:

✓ $H^\bullet_R(S, X) \cong H^\bullet(B(S), X)$;

✓ an explicit CW-complex, easy to define, hard to compute;

the only computations I am aware of are *Fenn–Rourke–Sanderson* '07:

1) trivial quandle $T_n = (\{1, \ldots, n\},\ a \lhd b = a)$: $B(T_n) \cong \Omega(\vee_n \mathbb{S}^2)$;

2) free rack on n generators FR_n: $B(FR_n) \cong \vee_n \mathbb{S}^1$;

✓ used to extract structural information on $H^\bullet_R(S, X)$, e.g. the cup product (even better: a Zinbiel product);

✓ $\pi_1(B(S)) \cong As(S)$,

where $As(S) := \langle S \,|\, a\,b = b\,(a \lhd b) \rangle$ is the associated group of $(S, \lhd)$.

8 Interpretations of rack cohomology

- ✓ classifying space;
- ✓ quantum shuffles;
- ✓ pre-cubical cohomology;
- ✓ shelf $\rightsquigarrow$ explicit d.g. bialgebra $\rightsquigarrow$ cohomology;
- ✓ ~~operads~~.

(*Serre ’51*, *Baues ’98*, *Clauwens ’11*, *Covez ’12*, *L. ’13*, *L. ’17*, *Covez–Farinati–L.–Manchon ’19*.)

9 Rack cohomology vs group cohomology

The associated group of $(S, \triangleleft)$:

$$\mathrm{As}(S) := \langle S \mid a\,b \;=\; b\,(a \triangleleft b)\rangle$$

Theorem (*Joyce* '82): One has a pair of adjoint functors

$$\mathrm{As} : \mathbf{Rack} \rightleftarrows \mathbf{Group} : \mathrm{Conj}\,.$$

Theorem (*García Iglesias & Vendramin* '16): For a finite indecomposable quandle S,

$$H^2_{\mathrm{R}}(S, X) \cong X \times \mathrm{Hom}(N(S), X).$$

Here $N(S)$ is a finite group (the stabilizer of an $a_0 \in S$ in $[\mathrm{As}(S), \mathrm{As}(S)]$).

Theorem (*Fenn–Rourke–Sanderson* '95): There is a graded algebra morphism $HH^\bullet(\mathrm{As}(S), X) \to H^\bullet_{\mathrm{R}}(S, X)$.

10 Betti numbers

Theorem (*Etingof–Graña* '03): If $(S, \lhd)$ is a rack and $\# \operatorname{Inn}(S) \in X^*$, then

$$\boxed{H_R^k(S, X) \cong X^{r^k}}$$

- ✓ $\operatorname{Orb}(S) = \{$ orbits of S w.r.t. $a \sim a \lhd b\}$, $r = \# \operatorname{Orb}(S)$;
- ✓ $\operatorname{Inn}(S)$ is the subgroup of $\operatorname{Aut}(S)$ generated by $t_b \colon a \mapsto a \lhd b$.

Bad news: If $\# \operatorname{Inn}(S) \in X^*$, then

quandle cocycle invariants = coloring invariants + linking numbers.

Hope: Look at $X = \mathbb{Z}_p$, or at the p-torsion of $H_R^k(S, \mathbb{Z})$, where $p \mid \# \operatorname{Inn}(S)$.

It works, and yields interesting invariants!

Homotopical tools: framework

Problem: Full rack/quandle (co)homology of a rack is hard to compute.

The only full computations I know of are:

- ✓ 1) trivial quandles;
- ✓ 2) free racks and quandles;
- ✓ 3) Alexander quandles of prime order (*Nosaka* '13).

So, new tools are necessary.

Theorem (*Szymik* '19): Quandle cohomology is a Quillen cohomology.

Applications:

- ✓ excision isomorphisms;
- ✓ Mayer–Vietoris exact sequences.

12 Homotopical tools: example

A permutation ϕ on a set S $\rightsquigarrow$ permutation rack $(S\,,\ a \lhd_{\phi} b = \phi(a))$.

Theorem (*L.–Szymik* '20): $\boxed{H^{R}_{k}((S, \lhd_{\phi}), X) \cong X^{\beta_k}}$ where

- ✓ $\beta_0 = 1,\ \beta_1 = r,\ \beta_{n+2} = (r-1)\beta_{n+1} + r_f \beta_n, \qquad n \geqslant 0;$
- ✓ $r = \#\{$ orbits of $\phi\}$, $\qquad r_f = \#\{$ finite orbits of $\phi\}$.

Remark: $H^{R}_{\bullet}(S, \lhd_{\phi})$ contains more information than $\mathrm{As}(S, \lhd_{\phi})$.

Sketch of proof:

Step 1 Explicit computations for free permutation racks
(= all orbits are infinite).

Trick: $H^{R}_{k} = \mathrm{Ker}\, d^{R}_{k} / \mathrm{Im}\, d^{R}_{k+1}$
study chains up to boundaries, then restrict to cycles
(usually: determine cycles, then mod out the boundaries).

Step 2 Choose a simplicial resolution by free permutations $F_\bullet \to S$

$\rightsquigarrow$ a double complex $E^0_{p,q} = C^R_q(F_p)$

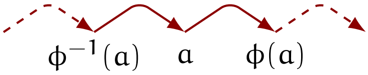
$\rightsquigarrow$ two spectral sequences with the same target.

Step 3 Computations in the spectral sequences:

1st SS: $E^\infty_{p,q} \cong \begin{cases} H^R_q(S) & \text{if } p = 0, \\ 0 & \text{if } p \neq 0. \end{cases}$

2nd SS: $E^2_{\bullet,q} \cong \overline{H}_\bullet(S /\!\!/ \phi)^{\otimes(q-1)} \otimes H_\bullet(S /\!\!/ \phi)$,

where $S /\!\!/ \phi$ is the homotopy orbit space:



$S /\!\!/ \phi = r_f$ circles $\bigsqcup$ $r - r_f$ lines

Step 4 For the 2nd SS, show that $E^\infty = E^2$.

For this, find enough independent elements in $H^R_q(S)$.