Quandle Homology is a Braided Homology

Victoria LEBED

I. Background: J. Przytycki's talk and beyond

1) Self-distributive structures $\xrightarrow{\text{tools}}$ knot theory

$(Q,\triangleleft)$-colorings of a knot: 

R III: 

unoriented knots / or. knots / braids / pos. braids:

- R III $\Leftrightarrow$ $\boxed{(a\triangleleft b)\triangleleft c=(a\triangleleft c)\triangleleft(b\triangleleft c)}$ (SD) — shelf
- R II $\Leftrightarrow$ $\exists\,\tilde{\triangleleft}$ s.t. $(a\triangleleft b)\tilde{\triangleleft} b=(a\tilde{\triangleleft} b)\triangleleft b=a$ (R) — rack
- R I $\Leftrightarrow$ $a\triangleleft a=a$ (Q)) — quandle
- orientation independance $\Leftrightarrow$ $(a\triangleleft b)\triangleleft b=a$ — kei

Ex: (group G, $a\triangleleft b=b^{-1}ab$) is a quandle.

Theory: Joyce, Matveev: The fundamental quandle of a knot defines it up to the symmetry $K\leftrightarrow -K^*$.

Practice: $\#\{Q\text{-colorings of }K\}$ is a knot invariant $\Leftarrow$ quandle invariant

$|| \ \sum_{Q\text{-col. of }K} 1$ $\qquad = \mathrm{Hom}_{Qu}(FQ(K),Q)$

gen^n $\downarrow$

$\sum_{Q\text{-col. of }K}\ \sum \pm\, \omega(a\otimes b)$ is a knot invariant iff $\Leftarrow$ quandle cocycle invariant

($\omega(a\otimes b)\in\mathbb{R}$: Boltzmann weight)

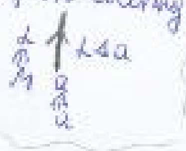
I) $\omega(a\otimes a)=0$

III) $\omega\big((b\otimes c-(a\triangleleft b)\otimes c-(a\triangleleft c)\otimes(b\triangleleft c))-(a\otimes b-a\otimes c+b\otimes c)\big)=0$

2 maps $V^{\otimes 3}\to V^{\otimes 2}$, $V=\mathbb{R}Q$

gen^{ns} $\swarrow$ K^n in S^{n+2} $\downarrow$ $V^{\otimes n+2}\to V^{\otimes n+1}$

$\downarrow$ $\sum\pm\,\omega(d\otimes a\otimes b)$ $\downarrow$ $M\otimes V^{\otimes 3}\to M\otimes V^{\otimes 2}$

region coloring: 

Rmk: quandle cocycle invariants can distinguish K from $-K^*$ (ex.: trefoil knot)

2) Parallel homology theories

SD structures | **associative structures**

Q: quandle, $V = \mathbb{R}Q$

M: Q-module

$(m \triangleleft a) \triangleleft b = (m \triangleleft b) \triangleleft (a \triangleleft b)$

V: unitary ass. algebra

M: V-module

$(m \cdot v) \cdot w = m \cdot (v \cdot w)$

presimplicial structure $\begin{cases} d_{(n),i} : M \otimes V^{\otimes n} \to M \otimes V^{\otimes n-1}, \ 1 \leq i \leq n \\ \text{s.t. } d_i \circ d_j = d_{j-1} \circ d_i \ \forall i < j \end{cases}$

$\Rightarrow d_{n-1} \circ d_n = 0,$

$d_n = \sum_{i=1}^{n} (-1)^{i-1} d_{n,i}$

$(m, a_1, \dots, a_n) \xmapsto{d_i} (m \triangleleft a_i, a_1 \triangleleft a_i, \dots, a_{i-1} \triangleleft a_i, a_{i+1}, \dots, a_n)$

$m \otimes v_1 \otimes \dots \otimes v_n \xmapsto{d_i} m \otimes v_1 \otimes \dots \otimes v_{i-1} v_i \otimes \dots \otimes v_n$

weakly simplicial structure $\begin{cases} s_{(n),i} : M \otimes V^{\otimes n} \to M \otimes V^{\otimes n+1}, \ 1 \leq i \leq n \\ \text{s.t. } d_j \circ s_i = \begin{cases} s_i \circ d_{j-1}, & j > i+1 \\ d_{i+1} \circ s_i, & j = i \\ s_{i-1} \circ d_j, & j < i \end{cases} \end{cases}$

$\Rightarrow (\sum_i \operatorname{Im} s_i, d)$ is a subcomplex of $(M \otimes T(V), d)$

$\xmapsto{s_i} (m, a_1, \dots, a_{i-1}, a_i, a_i, a_{i+1}, \dots, a_n)$

$\xmapsto{s_i} m \otimes v_1 \otimes \dots \otimes 1 \otimes v_i \otimes \dots \otimes v_n$

and many other properties...

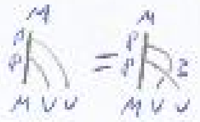
II. When parallel theories meet

1) Braided homology $\xleftarrow{\text{tools}}$ Braid theory

$(V, \sigma: V\otimes V \to V\otimes V)$: Braided vector space: $\sigma_1\circ\sigma_2\circ\sigma_1 = \sigma_2\circ\sigma_1\circ\sigma_2 : V^{\otimes 3}$ ⟲, $\sigma_1 = \sigma\otimes Id_V$, $\sigma_2 = Id_V\otimes\sigma$ (YBE)

Rmk: no invertibility condition

$(M, \rho: M\otimes V \to M)$: Braided V-module:



$(\underset{\text{Br. v.sp.}}{V,\sigma}, \Delta)$: left-braided coalgebra:

(Cocomm)

Rmk: 3 valent knotted graphs & handle-body knots

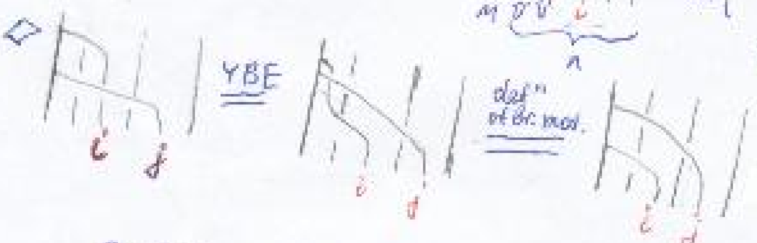
Thm (L., 2012): 1) $d_{n,i} :=$

$= \rho\circ\sigma_1\circ\sigma_2\circ\ldots\circ\sigma_{i-1}$ defines a presimplicial structure on $M\otimes T(V)$.

2) $s_{n,i} :=$

$= \Delta_i$ completes it into a weakly simplicial structure.

◇



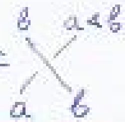
◆

Rmks: → right & 2-sided versions
→ functoriality
→ works in preadditive monoidal categories
→ quantum shuffles
→ sign in $\sum_i (-1)^{i-1} d_i$ is the intersection nb of the diagram

Related constructions:
→ Carter-Elhamdadi-Saito: hom. of the set-theoretic solutions to YBE
→ Majid: Braided-differential calculus: $d^2\neq 0$, ex: $V = \mathbb{R}x$, $\sigma(x\otimes x) = q\,x\otimes x$, $q\in\mathbb{R}^*$
$d(x^{\otimes n}) = \underbrace{\frac{q^n-1}{q-1}}_{"(n)_q"} x^{\otimes n-1}$
→ Eisermann: Yang-Baxter cochain complex $(Hom_{\mathbb{R}}(V^{\otimes n}, V^{\otimes n}))$

2) SD or ass. structure $\longmapsto$ braided v. sp

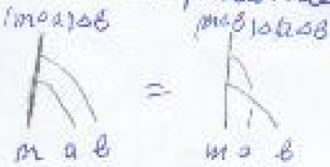
classical $\mapsto$ homology theory $\xleftarrow{\text{Thm}}$ braided v. sp

$\sigma_{SD} =$ 

YBE $\Leftrightarrow$ (SD) for ◁

$\exists \sigma^{-1} \Leftrightarrow$ (R)

br. mod. $\Leftrightarrow$ quandle mod.



$\Delta_{SD}: a \mapsto a \otimes a$

(cocomm) $\Leftrightarrow$ (Q)

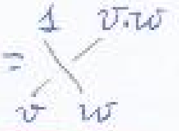
braided homologies

$\Downarrow$

→ 1-term distributive (Przytycki-Sikora)

→ rack (Fenn-Rourke-Sanderson)

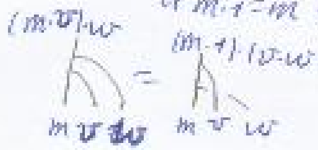
→ quandle (Carter-Jelsovsky-Kamada-Langford-Saito)

$\sigma_{Ass} =$ 

YBE $\Leftrightarrow$ ass-ty for ·
if $v \cdot 1 = v \ \forall v \in V$

$\exists \sigma^{-1}$

br. mod. $\Leftrightarrow$ algebra mod
if $m \cdot 1 = m \ \forall m \in M$



$\Delta_{Ass}: v \mapsto 1 \otimes v$

(cocomm) $\Leftrightarrow 1 \cdot v = v$

braided homologies

$\Downarrow$

→ bar

→ Hochschild

Leibniz algebras

$\sigma_{Lei}: v \otimes w \mapsto w \otimes v + 1 \otimes [v, w]$

(YBE) $\Leftrightarrow [[v,w],u] = [v,[w,u]] + [[v,u],w]$ (Lei)
if $[v,1] = [1,v] = 0$

$\exists \sigma^{-1}$

br. mod. $\Leftrightarrow$ Leibniz mod.

$\Delta_{Lei}: v \mapsto 1 \otimes v + v \otimes 1,\ v \in V'$
$1 \mapsto 1 \otimes 1$

if V is split: $V = V' \oplus \mathbb{R}1$,
$V' \subset V$ Leibniz subalgebra

braided homologies

$\Downarrow$

→ Leibniz:

Leibniz	$T(V)$	Leibniz
anti-symmetrizn → ↓	↓	↓
Lie	$\Lambda(V)$	Chevalley-Eilenberg