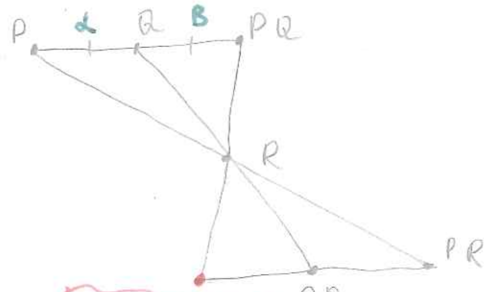
How forgetting group laws leads to a universal knot invariant

Victoria Lebed,
Galway
29/09/16

I) Motivations & definitions

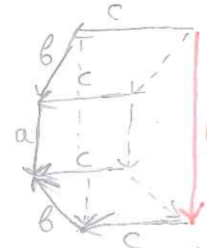
Gavin Wraith's school-time riddles:

① 

$\boxed{(PQ)R = (PR)(QR)}$

② $a, b, c \in S_5$

$a \triangleleft b = b^{-1} a b$



$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$

$\boxed{R3}$ (or self-distributivity)

Definition: $(A, \triangleleft)$ is called
(Wraith & Conway '59) — A: set, $\triangleleft$: binary operⁿ

- **shelf** if R3
- **(w)rack** if R3 & R2
- **quandle** if R3 & R2 & R1.

$\boxed{R2}$ $\forall b \in A$, the right translatⁿ $a \mapsto a \triangleleft b$ is a bijection $A \xrightarrow{\sim} A$

$\boxed{R3}$ $\forall a \in A$, $a \triangleleft a = a$.

aba^{-1}) Examples

① Alexander quandle: $A \in \mathbb{Z}[t^{\pm1}]$ Mod, $a \triangleleft b = ta + (1-t)b$.

② Conjugatⁿ quandle: group G, $a \triangleleft b = b^{-1}ab$.

Rmk: free quandles come from these.

③ Core quandle: group G, $a \triangleleft b = ba^{-1}b$.

④ Coxeter rack: $V = k^m$ & a "good" form ⇝ $A = V \setminus \{0\}$, $a \triangleleft b = a - 2\frac{(a,b)}{(b,b)} b$.

- bilin.
- sym.
- non-deg.

$a \triangleleft a = -a$.

⑤ Free shelf on a ⇝ a total order on braid groups (Dehornoy '91)

⑥ Free shelf on a / $\underbrace{a \triangleleft (a \triangleleft (a \triangleleft a))}_{2^n+1\ a\text{'s}} = a$ is a finite shelf, related to large cardinals.

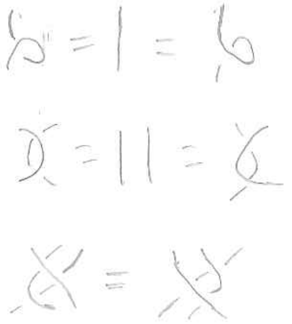
$\because$ Laver's n-table '95

8) Knot invariants

Reidemeister '27: $\boxed{\text{Knots} = \text{Diagrams} / {}^{\text{moves}}_{R1-R3}}$

Ex.: trefoil knot

R 1: , R 2: , R 3:



$Q_T = FQ\langle a,b,c\rangle \Big/ \begin{matrix} a = c \triangleleft b, \\ b = a \triangleleft c, \\ c = b \triangleleft a \end{matrix}$

free quandle

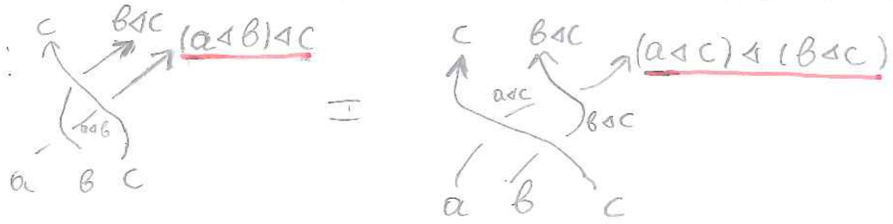
$\simeq FQ\langle a,b\rangle \Big/ \begin{matrix} a = (b \triangleleft a) \triangleleft b, \\ b = (a \triangleleft b) \triangleleft a \end{matrix}$

Knot $K \rightsquigarrow$ diagram $D_K \rightsquigarrow$ quandle Q_K (knot quandle)

- generators = arcs
- relations: $a \; \uparrow_b \; a \triangleleft b$ $\leftsquigarrow$ (cf. Wirtinger presentat^n of the knot group)

Prop.: Q_K does not depend on the choice of the diagram D_K.

□ R3:



⊠

Thm (Joyce, Matveev '82): Q_K is a weak universal knot invariant:

$Q_K \simeq Q_{K'} \Rightarrow K = K'$ or $-K^*$ ← mirror

opposite orient^n

Rmk: $UEG(Q_K) \simeq \pi_1(\mathbb{R}^3 \setminus K)$

universal enveloping group; the knot group.

Pb: Quandles are difficult to compare.

Solut^n: Consider A-colourings of K, i.e. $\mathrm{Rep}(Q_K, A)$

some well-understood quandle

$$n_{K,A} \in \mathbb{Z} \xleftarrow{\#} \mathrm{Rep}(Q_K, A) \longrightarrow \text{more information? see next part!}$$

Ex.: $A = \mathbb{Z}_3$, $a \triangleleft b = 2b - a$

$a \; \uparrow_b \; a \triangleleft b$ either $a = b = a \triangleleft b$, or they are all different.

$\rightsquigarrow$ Fox colourings 50's

$n_{O,A} = 3$

$n_{T,A} = 3 + 3! = 9$

$\Rightarrow$ Trefoil knot $\neq$ Unknot

d^2) Quandle cohomology.

Fenn-Rourke-Sanderson '95, Graña '00, Carter-Jelsovsky-Kamada-Langford-Saito '03
preprints '99

Rack cohomology of a rack A is the cohomology of the complex

$$C^n_R(A, \mathbb{Z}) = Map(A^{\times n}, \mathbb{Z})$$

or any abelian group
or twisted coeffs

$$\boxed{d^n f(a_1, \dots, a_{n+1}) = \sum_{i=1}^{n+1} (-1)^i \left[f(\dots, \hat{a}_i, \dots) - f(a_1 \triangleleft a_i, \dots, a_{i-1} \triangleleft a_i, a_{i+1}, \dots, a_{n+1}) \right]}$$

If A is a quandle, then $C^n_Q(A, \mathbb{Z}) = \{ f : A^{\times n} \to \mathbb{Z} \mid f(\dots, a, a, \dots) = 0 \}$ is a subcomplex. Its cohomology is called the **quandle cohomology** of A.

Applications:

[1] For $\omega \in H^2_Q(A, \mathbb{Z})$, the multiset $\{ BW_\omega(C) \mid C \in Rep(Q_K, A) \}$ is a knot invariant.
|| $B_{K,A,\omega}$

$$BW_\omega(C) = \sum \omega(a, b) - \sum \omega(a, b)$$

↑ Boltzmann weight

Rmk: • $B_{K,A,0} = \{ \underbrace{0, \dots, 0}_{n_{K,A} \text{ times}} \}$

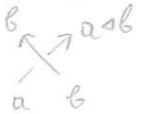
- This invariant can distinguish K from $-K^*$.
- $\omega \in H^K_Q(A, \mathbb{Z}) \rightsquigarrow$ invariant of $(K-1)$-dimensional knots in $\mathbb{R}^{K+1}$.

[2] Computation of $H^2_R(A, \mathbb{Z})$ for certain racks is an important step in classifying pointed Hopf algebras.

X) Yang-Baxter equation

Rack $A \rightsquigarrow \sigma : A \times A \to A \times A$ solution to the YBE $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 : A^{\times 3} \to A^{\times 3}$

$(a,b) \mapsto (b, a \triangleleft b)$ $\quad\quad \sigma_1 = \sigma \times Id_A,\ \sigma_2 = Id_A \times \sigma.$

$\sigma =$ 

$\Updownarrow$

R3 move.

Other sources of solutions to the YBE:

A) monoid $\rightsquigarrow \sigma(a,b) = (1, ab)$. $\quad$ YBE $\Leftrightarrow$ associativity

B) Lie algebra $\rightsquigarrow \sigma(a \otimes b) = 1 \otimes [a,b] + b \otimes a$ $\quad$ YBE $\Leftrightarrow$ Jacobi relation

C) factorised group $G = HK$ $\rightsquigarrow \sigma(g_1, g_2) = (\underset{H}{\underset{\Uparrow}{h}}, \underset{K}{\underset{\Uparrow}{k}})$ $\quad g_1 g_2 = hk$

D) lattice $\rightsquigarrow \sigma(a,b) = (\min\{a,b\}, \max\{a,b\})$.

AND MANY MORE !!!

L-Vendramin '16: $\boxed{\text{a "nice" set-theoretic solution } \sigma \text{ to the YBE} \longmapsto \text{a shelf that captures important properties of } \sigma}$

e.g. the associated representations of the braid groups B_n.

Carter-Elhamdadi-Saito '04, L.'13: $\boxed{\text{a (co)homology theory for general YBE solutions}}$

Applications:

- unifies basic (co)homology theories: → rack & quandle → group, Hochschild → Chevalley-Eilenberg ...
- guides in developing (co)homology theories for new algebraic structures
- graphical calculus $\quad$ • additional structure (cup product etc.)
- knot invariants
- YBE solution $(A, \sigma) \mapsto UEG(A,\sigma) = \langle A \mid ab = b'a' \text{ whenever } \sigma(a,b) = (b', a') \rangle$ (universal enveloping group)

$$\boxed{H^n(A,\sigma) \xleftarrow{QS} H^n(UEG(A,\sigma), \mathbb{Z})}$$

smaller complexes $\quad\quad$ more tools available.

QS (the quantum symmetriser) is an isomorphism when

* $\sigma^2 = Id$, coeffs in $\mathbb{Q}$ (Farinati & García-Galofre '16)
* $\sigma^2 = \sigma$ (L.'16; proof: algebraic discrete Morse theory) (cf. examples A, C, D above + Young tableaux).