YBE SOLUTIONS OF PENTAGONAL TYPE

Paola Stefanelli – Università del Salento

Algebra Days in Caen 2022: from Yang–Baxter to Garside, March 24 – 25 2022

A brief overview

The Yang-Baxter and pentagon equations are two basic equations of Mathematical Physic that lie in the class of polygon equations [4]. Given a set S, a map $s : S \times S \to S \times S$ is said to be a *set theoretical solution of the quantum Yang-Baxter equation*, or shortly QYBE solution, if

$$s_{23}\, s_{13}\, s_{12} = s_{12}\, s_{13}\, s_{23},$$

with $s_{12} = s \times \mathrm{id}_S$, $s_{23} = \mathrm{id}_S \times s$, and $s_{13} = (\mathrm{id}_S \times \tau)\, s_{12}\, (\mathrm{id}_S \times \tau)$ where τ is the flip map. Instead, s is called a *set-theoretical solution of the pentagon equation*, or shortly PE solution, if

$$s_{23}\, s_{13}\, s_{12} = s_{12}\, s_{23}.$$

Determining all QYBE solutions is an open question posed by Drinfel'd [5] around the 90s.

In [2], we focus on maps satisfying both the equations, which we call *solutions of the quantum Yang-Baxter equation of pentagonal type*, or briefly, P-QYBE solutions. In particular, we give a description of such maps and study their behaviour. Contrary to what happens in general, we prove that among these maps there are some whose powers are solutions.

Basics on the pentagon equation

According to the notation introduced in [1], given a set S and a map s from $S \times S$ into itself, we write

$$s(a,b) = (ab,\, \theta_a(b)),$$

where θ_a is a map from S into itself, for every $a \in S$.

Proposition 1 The map $s(a,b) = (ab,\, \theta_a(b))$ is a PE solution on S if and only if the following conditions hold

$$(ab)c = a(bc) \tag{1}$$
$$\theta_a(b)\theta_{ab}(c) = \theta_a(bc) \tag{2}$$
$$\theta_{\theta_a(b)}\theta_{ab} = \theta_b \tag{3}$$

for all $a, b, c \in S$.

Clearly, the condition (1) leads to consider semigroups. For instance, if S is a semigroup and $f \in End(S)$, with $f^2 = f$, then the map $s(a,b) = (ab, f(b))$ is a PE solution on S. In particular, if S is a group, the only invertible PE solution s on S is given by $s(a,b) = (ab, 1)$ (see [6]).

A complete description of not necessarily bijective PE solutions on groups can be found in [1]. Later, a description of all involutive PE solutions has given in [3].

P-QYBE solutions

Definition Let S be a set and $s(a,b) = (ab, \theta_a(b))$ a PE solution on S. The map s is said to be a P-QYBE solution if it is also a QYBE solution.

Proposition 2 Let S be a semigroup and $s(a,b) = (ab, \theta_a(b))$ a PE solution on S. Then, the map s is a QYBE solution if and only if

$$abc = a\theta_b(c)bc \tag{Y1}$$
$$\theta_a\theta_b = \theta_b \tag{Y2}$$
$$\theta_a(bc) = \theta_{\theta_b(c)}(bc) \tag{Y3}$$

are satisfied, for all $a, b, c \in S$.

Proposition 3 Let $s(a,b) = (ab, \theta_a(b))$ be a P-QYBE solution on a semigroup S. Then, the following conditions hold:

1. the map θ_a is idempotent, for every $a \in S$;
2. $\theta_{a|_{S^2}} = \theta_{b|_{S^2}}$, for all $a, b \in S$;
3. if $S^2 = S$, then $s(a,b) = (ab, f(b))$, where f is an idempotent endomorphism of S.

Examples

1. The map $s\,(a,b) = (ab, e)$, with e a left identity (or a right identity) for a semigroup S. In particular, if S is a group, the unique P-QYBE solution is $s(a,b) = (ab, 1)$.
2. The map $s\,(a,b) = (ab, b)$, with S is a left quasi-normal semigroup, i.e., $abc = acbc$, for all $a, b, c, \in S$.
3. The map $s\,(a,b) = \left(ab, b^{-1}b\right)$, with S a Clifford semigroup.

Now, we focus on semigroups S that belong to the variety

$$\mathcal{S} := [abc = adbc]$$

which immediately ensures (Y1) (see [7]). In this way, one has to find maps θ_a from S into itself satisfying just (Y2) and (Y3).

Proposition 4
Let $S \in \mathcal{S}$ such that $S^2 = S$. Then, the unique P-QYBE solutions on S are $s(a,b) = (ab, f(b))$, with f an idempotent endomorphism of S.

Examples

1. If $S \in \mathcal{S}$, the map $s\,(a,b) = (ab, bab)$.
2. The map $s(a,b) = (f(a), g(b))$, where $ab = f(a)$, for all $a, b \in S$.

Powers of P-YBE solutions

We recall that a map s is a QYBE solution if and only if the map $r := \tau s \tau$ is a solution of the braid equation, i.e., it holds

$$r_{12}\, r_{23}\, r_{12} = r_{23}\, r_{12}\, r_{23}.$$

In this case, we say that r is a YBE solution.

Note that if r is a YBE solution, its n-th power r^n is not necessarily a YBE solution.

If s is a P-QYBE solution, we say that the corresponding map $r := \tau s \tau$ is a P-YBE solution.

Theorem 1 Let $S \in \mathcal{S}$ and $r(a,b) = (\theta_a(b), ab)$ a P-YBE solution on S. Then, it holds $r^5 = r^3$ and the powers r^2, r^3, r^4 of the map r are still YBE solutions. In addition, if S is idempotent, it holds $r^4 = r^2$.

There exist P-YBE solutions r for which $r^5 = r^3$, but the powers of r are not solutions. The P-YBE solution $r(a,b) = (b, ab)$ defined on a left quasi-normal semigroup S is such an example.

Example The P-YBE solution $r\,(a,b) = (bab, ab)$ on $S \in \mathcal{S}$ satisfies $r^4 = r^2$ and the powers

$$r^2\,(a,b) = (aab,)$$
$$r^3\,(a,b) = (bab, aab)$$

are YBE solutions.

References

[1] F. Catino, M. Mazzotta, and M. M. Miccoli. Set-theoretical solutions of the pentagon equation on groups. *Comm. Algebra*, 48(1):83–92, 2020.

[2] F. Catino, M. Mazzotta, and P. Stefanelli. Set-theoretical solutions of the Yang–Baxter and pentagon equations on semigroups. *Semigroup Forum*, 100(3):1–26, 2020.

[3] Ilaria Colazzo, Eric Jespers, and Ł ukasz Kubat. Set-theoretic solutions of the pentagon equation. *Comm. Math. Phys.*, 380(2):1003–1024, 2020.

[4] A. Dimakis and F. Müller-Hoissen. Simplex and Polygon Equations. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 11:Paper 042, 49, 2015.

[5] V. G. Drinfel′ d. On some unsolved problems in quantum group theory. In *Quantum groups (Leningrad, 1990)*, volume 1510 of *Lecture Notes in Math.*, pages 1–8. Springer, Berlin, 1992.

[6] R. M. Kashaev and S. M. Sergeev. On pentagon, Ten-Term, and Tetrahedron Relations. *Comm. Math. Phys.*, 195(2):309–319, 1998.

[7] R. A. R. Monzo. Pre-compatible almost endomorphisms and semigroups whose cube is a band. *Semigroup Forum*, 67(3):355–372, Sep 2003.

Contact Information

paola.stefanelli@unisalento.it