Structure groups of racks & quandles

Lebed
Oberwolfach
2019

1 Quandles, YBE solutions & groups

Data: Set S & binary operation $\triangleleft$.

Axioms: (1) $\boxed{(a\triangleleft b)\triangleleft c=(a\triangleleft c)\triangleleft(b\triangleleft c)}$ (self-distributivity)

(2) $\forall a,\ t_a: x\mapsto x\triangleleft a$ is a bijection $S\to S$

(3) $a\triangleleft a=a$

Terminology for $(S,\triangleleft)$:
(1) $\Leftrightarrow$ shelf $\Leftrightarrow \forall a,\ t_a\in \mathrm{End}(S,\triangleleft)$
(1)&(2) $\Leftrightarrow$ rack $\Leftrightarrow \forall a,\ t_a\in \mathrm{Aut}(S,\triangleleft)$
(1)&(3) $\Leftrightarrow$ spindle
(1)&(2)&(3) $\Rightarrow$ quandle

$\boxed{(S,\triangleleft)\rightsquigarrow(S,r_\triangleleft)}$, $\boxed{r_\triangleleft(a,b)=(b,a\triangleleft b)}$ "semi-trivial" solution

(1) $\Leftrightarrow$ YBE solution
always left non-deg.
(2) $\Leftrightarrow$ right non-deg.
$\Leftrightarrow$ invertible
(1) $\Leftrightarrow$ square-free $\Leftrightarrow \exists t: S\xrightarrow{1:1}S$ s.t.
$r(a,a)=(a,a)$ $\qquad r(t(a),a)=(t(a),a)$

Examples:

S	$a\triangleleft b$	name
group	$b^{-1}ab,\ ba^{-1}b,\dots$	conjugation quandle
$\mathbb{Z}[t^{\pm1}]$ Mod	$ta+(1-t)b$	Alexander quandle
$\mathbb{Z}$ or $\mathbb{Z}_n$	$a+1$	cyclic rack

$(S,\triangleleft)\rightsquigarrow(S,r_\triangleleft)\rightsquigarrow \boxed{G(S,r_\triangleleft)=\langle a,\ a\in S \mid ab=ba^b,\ a,b\in S\rangle}$ $\qquad a^b:=a\triangleleft b$

$\quad\ \ \| $
$\boxed{G(S,\triangleleft)}$ structure group of $(S,\triangleleft)$
$\qquad\qquad\qquad \|$
adjoint = associated = enveloping = ...

Adjoint functors: ① $\boxed{\text{Quandle} \underset{\text{Conj}}{\overset{G}{\rightleftarrows}} \text{Group}}$

② $\boxed{\text{Rack} \overset{Q}{\rightleftarrows} \text{Quandle}}$ $\qquad Q(S,\triangleleft):=\left(S/_{\forall a,\ a\sim a\triangleleft a},\ \bar{\triangleleft}\right)$

$$\begin{array}{ccc} \text{Rack} & \xrightarrow{Q} & \text{Quandle} \\ & \searrow \quad \swarrow & \\ & \text{Group} & \end{array}$$

induced operation
- well-defined
- a quandle operation

So from now on we concentrate on structure groups of finite quandles

[2] Structure groups of quandles are almost free abelian (L.-Vendramin '19)

$\Gamma := \# \mathrm{Orb}(S,\triangleleft)$

orbits w.r.t. all t_a's

$\mathbb{Z}^\Gamma \simeq \bigoplus_{i \in \mathrm{Orb}(S,\triangleleft)} \mathbb{Z} e_i$

$G(S,\triangleleft)$ is quite close to $\mathbb{Z}^\Gamma$:

① $G(S,\triangleleft) \overset{p}{\twoheadrightarrow} Ab(G) \twoheadrightarrow \mathbb{Z}^\Gamma$

$a \longmapsto e_{\mathrm{orb}(a)}$

② $0 \to \mathbb{Z}^\Gamma \xrightarrow{\iota} G(S,\triangleleft) \xrightarrow{\pi} \underbrace{\overline{G}(S,\triangleleft)}_{\text{finite group}} \to 0$

$e_i \mapsto a^d$ — any a with $\mathrm{orb}(a) = i$

$p \circ \iota = d\, \mathrm{Id}_{S}$ (*)

$d := \min\{k \mid \forall a,\ t_a^k = \mathrm{Id}_S\}$.

class of $(S,\triangleleft)$

Ex.: $d = 1 \Leftrightarrow$ trivial quandle: $a \triangleleft b = a$.

$d = 2 \Leftrightarrow (a \triangleleft b) \triangleleft b = a$ (Takasaki's keis)

Frozen elements: $a^d \in G(S,\triangleleft)$.

- $a \underset{\text{same orbit}}{\sim} b \Rightarrow a^d = b^d$ in G
- $a^d \in Z(G)$.

Cr: 1) G is virtually free abelian $\Rightarrow$ linear $\Rightarrow$ residually finite.

2) a finitary injectivity criterion:

$$\begin{tikzcd} G \arrow[r, two heads] & \overline{G} \\ S \arrow[u, "i"] \arrow[ur, "\overline{i}"'] \end{tikzcd} \qquad a \mapsto a$$

i injective
$\Updownarrow$
$\overline{i}$ injective
$\Updownarrow$
S is a subquandle of a finite conj^n quandle

3) $\Leftrightarrow$:
a) G free abelian
b) G abelian
c) G torsion-less
d) $G \simeq \mathbb{Z}^\Gamma$
e) p is an iso
f) G bi-orderable
g) G left-orderable

} G boring!

□ $a \Rightarrow f \Rightarrow g \Rightarrow c$
$\Leftarrow d \Leftarrow e$, $d \Rightarrow b \Rightarrow e$

$\Downarrow$ • p not iso $\Rightarrow \exists g \in G,\ g \neq 1,\ p(g) = 0$

$\overset{(*)}{\Rightarrow} g \notin \mathrm{Im}\,\iota = \ker \pi$

$\Rightarrow \pi(g) \neq 1,\ \exists k$ s.t. $\pi(g)^k = 1$ ($\overline{G}$ is finite)

$\Rightarrow g^k \in \ker \pi = \mathrm{Im}\,\iota$

$\Rightarrow g^k = \iota(x)$, $\underbrace{p \circ \iota(x)}_{= d \cdot x} = p(g^k) = k\, p(g) = 0 \Rightarrow x = 0 \Rightarrow g^k = 1$ ⊠.

3 Classification of finite quandles with boring structure group

(L.-Mortier '19)

{ In the quandle world, generic examples are boring, but interesting examples do exist!

Step 1: $\boxed{\mathcal{G}(S,\triangleleft) \text{ abelian} \Rightarrow (S,\triangleleft) \text{ abelian}}$: $\boxed{(a\triangleleft b)\triangleleft c = (a\triangleleft c)\triangleleft b}$

(equivalently, $\boxed{a\triangleleft(b\triangleleft c) = a\triangleleft b}$)

□ $S \curvearrowleft \mathcal{G}(S,\triangleleft)$ $\Rightarrow (a\triangleleft b)\triangleleft c = (a\cdot b)\cdot c = a\cdot bc = a\cdot cb = (a\cdot c)\cdot b = (a\triangleleft c)\triangleleft b$ ⊠.

$a\cdot b := a\triangleleft b$

So, the pb is reduced to a study of commuting permutations.

Step 2: Parametrise abelian quandles by matrices $M(S,\triangleleft)$. (recover Stanovsky et al. '15)

$\rightarrow \frac{r^2(r-1)}{2}$ parameters $\in \mathbb{N}$,

subject to some inequalities

Step 3: $(S,\triangleleft)$ abelian $\Rightarrow$ $\{0 \to \mathcal{G}' := [\mathcal{G},\mathcal{G}] \to \mathcal{G} \xrightarrow{p} \mathbb{Z}^r \to 0\}$

- finite ab. group, central in $\mathcal{G}$
- presentation matrix $M'(S,\triangleleft)$ explicit

Step 4: $\mathcal{G}$ abelian $\Leftrightarrow$ $\mathcal{G}'$ trivial $\Leftrightarrow$ $\gcd(\text{max. minors of } M') = 1$.

Application: YBE solution (S,r) (invertible non-deg.) $\rightsquigarrow$ structure rack $(S,\triangleleft_r)$ & group 1-cocycle $\{\mathcal{G}(S,r) \xrightarrow{1:1} \mathcal{G}(S,\triangleleft_r)\}$.

So, if $(S,\triangleleft_r)$ is as above, one gets a group 1-cocycle

$\mathcal{G}(S,r) \xrightarrow{1:1} \mathbb{Z}^k$, $k = \#\mathrm{Orb}(S,\triangleleft_r)$,

and extends to such r some results known for involutive r.

Question: $(S,\triangleleft_r)$ abelian $\Leftrightarrow$ r is ?

$\mathcal{G}(S,\triangleleft_r)$ abelian $\Leftrightarrow$ r is ?

Examples:

- $\mathcal{G}(S,\triangleleft) \simeq \mathbb{Z} \Leftrightarrow \#S = 1$
- $\mathcal{G}(S,\triangleleft) \simeq \mathbb{Z}^2 \Leftrightarrow (S,\triangleleft) \cong U_{m,n} = \{x_0,\dots,x_{m-1}, y_0,\dots,y_{n-1}\}$ $m \leq n$, $(m,n)=1$

$x_i \triangleleft x_j = x_i$, $y_k \triangleleft y_l = y_k$,

$x_i \triangleleft y_k = x_{i+1}$, $y_k \triangleleft x_i = y_{k+1}$

Bardakov–Nasybullov '19.

$\boxed{4}$ A curious property of $G(X, \triangleleft)$ (Ryder's thesis '93)

Recall: Quandle $\underset{\text{Conj}}{\overset{G}{\rightleftarrows}}$ Group

For a group Γ, put $\tilde{\Gamma} := G(\mathrm{Conj}(\Gamma)) = \langle e_g,\ g \in \Gamma \mid e_g e_h = e_h e_{h^{-1}gh},\ g,h \in \Gamma \rangle$

$$\boxed{0 \to K \to \tilde{\Gamma} \underset{s}{\overset{p}{\rightleftarrows}} \Gamma \to 0}$$

$\underset{\text{in } \tilde{\Gamma}}{\text{central}}$ $\quad e_g \mapsto g$

$e_g \leftarrow g$

s is a set-theoretic section only: $\underbrace{e_{gh}}_{\deg=1} \neq \underbrace{e_g e_h}_{\deg=2}$,

Define $\theta: \Gamma \times \Gamma \to K$ by $\boxed{\theta(g,h) = e_{gh}^{-1} e_g e_h}$ $\in K$.

It is a group 2-cocycle: $\theta(gh,k)\theta(g,h) = \theta(g,hk)\theta(h,k)$,

and $\boxed{\tilde{\Gamma} \simeq \Gamma \times_\theta K}$ (abelian extension)

$(g,k)\cdot(g',k') = (gg', kk'\theta(g,g'))$

Now, take $\Gamma = G(\underbrace{S, \triangleleft}_{\text{quandle}}) = \langle a,\ a \in S \mid ab = ba^b,\ a,b \in S \rangle$.

Then $s': \Gamma \to \tilde{\Gamma}$, $a \mapsto e_a$ defines a group section of p $\Rightarrow$ $\boxed{\tilde{\Gamma} \simeq \Gamma \times K}$

Ryder's conjecture: Γ is the structure group of a quandle

$$\Updownarrow$$

$\tilde{\Gamma} \simeq \Gamma \times K$ for an abelian group K.

Question: Any other reasonable characterisations of the structure groups of quandles?