Unexpected facets of the Yang-Baxter equation

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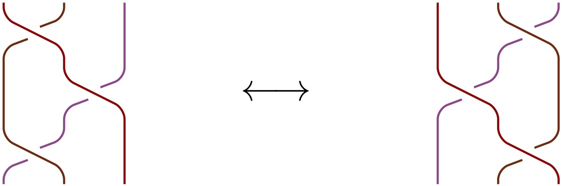
University of Nantes

Utrecht, September 29, 2015



1 Yang-Baxter equation

✓ A vector space V (or an object in any monoidal category)

✓ $\sigma\colon V^{\otimes 2} \to V^{\otimes 2}$

Yang-Baxter equation (YBE):

$$\sigma_1 \circ \sigma_2 \circ \sigma_1 = \sigma_2 \circ \sigma_1 \circ \sigma_2 \colon V^{\otimes 3} \to V^{\otimes 3}$$

where $\sigma_i = \mathrm{Id}_V^{\otimes i-1} \otimes \sigma \otimes \mathrm{Id}_V^{\otimes \cdots}$.

A map σ satisfying YBE is a braiding.

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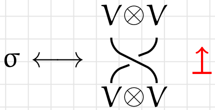

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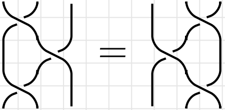

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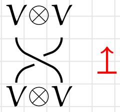

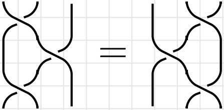

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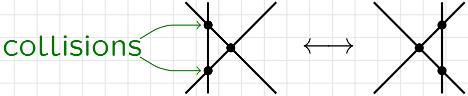
History:

→ **Particle physics**: factorization cond. for the dispersion matrix in the 1-dim. n-body problem (*McGuire, Yang, 60'*).



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History:

→ **Particle physics**: factorization cond. for the dispersion matrix in the 1-dim. n-body problem (*McGuire, Yang, 60'*).

→ **Statistical mechanics**: partition function for exactly solvable lattice models (*Baxter, 70'*), quantum inverse scattering method for completely integrable systems (*Faddeev et al., 1979*).

→ **Field theories**: factorizable S-matrices in 2-dim. quantum field theory (*Zamolodchikov, 1979*), conformal field theory.

→ **Quantum groups** (*Drinfel′d, 80'*).

→ C^* **algebras** (*Woronowicz, 80'*).

→ **Low-dimensional topology**.

....

2 A homology theory for the YBE

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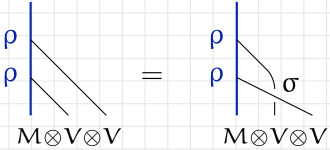
Ingredients:

✓ A braided vector space (V, σ);

✓ a left braided V-module: $(M, \rho\colon M \otimes V \to M)$ s.t.

$$\rho \circ \rho_1 = \rho \circ \rho_1 \circ \sigma_2 : \quad M \otimes V \otimes V \to M$$



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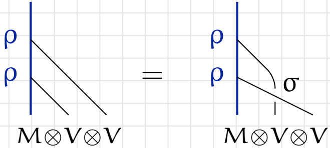

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Theorem (L. 2013): $M \otimes T(V) \otimes N$ carries a family of differentials $\delta^{(\alpha,\beta)} = \alpha\, {}^{\bullet}\delta + \beta\, \delta^{\bullet}$, $\alpha, \beta \in \mathbb{k}$.

(I.e., $\delta^{(\alpha,\beta)} \circ \delta^{(\alpha,\beta)} = 0$.)

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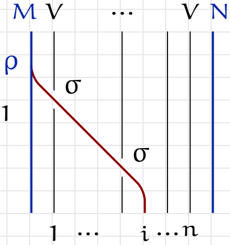
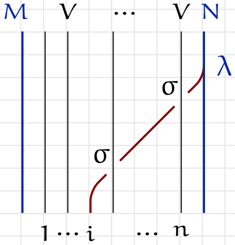
${}^{\bullet}\delta = \sum (-1)^{i-1}$ 

$\delta^{\bullet} = \sum (-1)^{i-1}$ 

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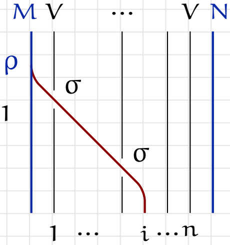

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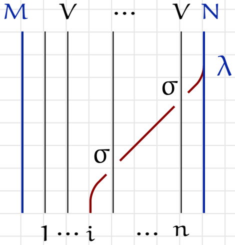
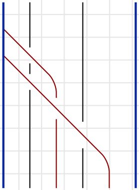
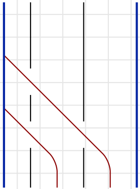


Proof:



$\overset{\text{YBE}}{=}$

$\overset{\text{br. mod.}}{=}$



& sign $= (-1)^{\#\text{cross.}}$

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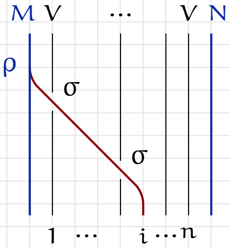

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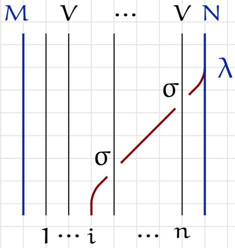

<u>Remarks</u>:

✓ Functoriality.

✓ Interpretation in terms of quantum shuffles (Rosso, 1995).

✓ Duality $\rightsquigarrow$ a cohomology theory.

✓ Pre-cubical structure.

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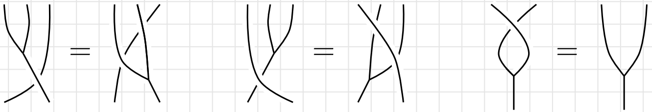
Braided coalgebra: br. v. sp. (V, σ) & $\Delta \colon V \to V \otimes V$ s.t.



(Cf. Reidemeister moves for knotted 3-valent graphs!)

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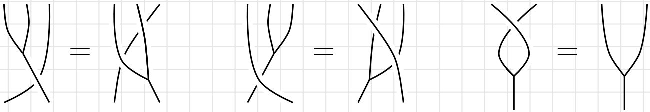

(Cf. Reidemeister moves for knotted 3-valent graphs!)

Theorem (L. 2013): All $\delta^{(\alpha,\beta)}$ restrict to $\sum_i \mathrm{Im}(\Delta_i)$.

$\rightsquigarrow$ normalization

3 Alg. structures via braidings

(A) Associative algebras

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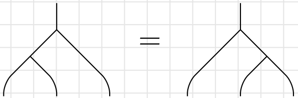
$(V, \cdot, 1)$ s.t.

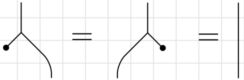
Associativity:

$(u \cdot v) \cdot w = u \cdot (v \cdot w)$



Unit axiom:

$1 \cdot v = v \cdot 1 = v$



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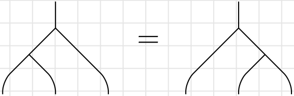

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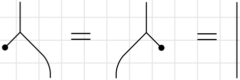
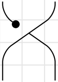


$\rightsquigarrow$ "Associativity braiding"

$$\sigma_{Ass} : v \otimes w \mapsto 1 \otimes v \cdot w$$



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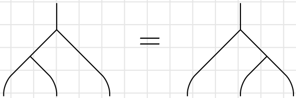

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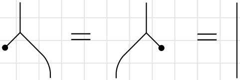

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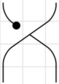

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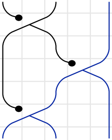
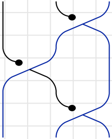


$=$



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✓ Braided homologies for (V, σ_{Ass}) include

→ bar; → Hochschild; → group.

3 Alg. structures via braidings

Ⓑ Leibniz algebras

$(V, [], 1)$ s.t.

Leibniz identity: $[v,[w,u]] = [[v,w],u] - [[v,u],w]$;

Lie unit axiom: $[1,v] = [v,1] = 0$.

(*Bloh 1965, Loday & Cuvier 1991*: a non-commutative generalization of Lie algebras.)

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$\rightsquigarrow$ "Leibniz braiding" $\sigma_{Lei} : v \otimes w \mapsto w \otimes v + 1 \otimes [v,w]$

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✓ A fully faithful functor $\mathbf{Lei} \hookrightarrow \mathbf{Br_\bullet}$.

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✓ σ_{Lei} is invertible.

✓ Braided mod. for $(V, \sigma_{Lei}) \longleftrightarrow$ anti-symmetric Leibniz mod. for $(V, [], 1)$.

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✓ Suppose that $V = V' \oplus \Bbbk 1$, $[V', V'] \subseteq V'$. Then $\boxed{\Delta_{\mathrm{Lei}} : v \mapsto 1 \otimes v + v \otimes 1, v \in V'; \quad 1 \mapsto 1 \otimes 1}$ turns $(V, \sigma_{\mathrm{Lei}})$ into a braided coalgebra.

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✓ Braided homologies for (V,σ_{Lei}) include Leibniz homology.

$$(M\otimes T(V), d_{Lei}) \qquad \text{Cuvier-Loday}$$

$$\downarrow \text{anti-symm.}$$

$$\textbf{Lie} \qquad V \longmapsto (M\otimes \Lambda(V), d_{CE}) \qquad \text{Chevalley-Eilenberg}$$

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✓ Explains the choice of the lift of the Jacobi identity.

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C Self-distributive structures

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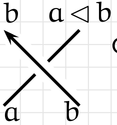
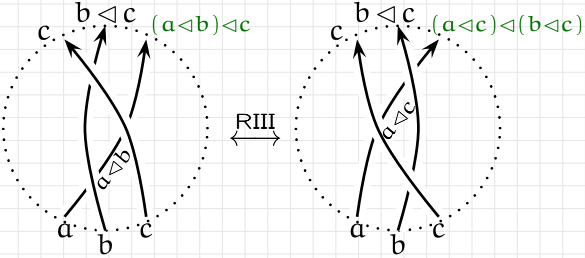


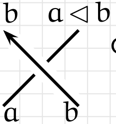
colorings by $(S, \triangleleft)$



RIII $\longleftrightarrow$ $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$ (SD)

3 Alg. structures via braidings

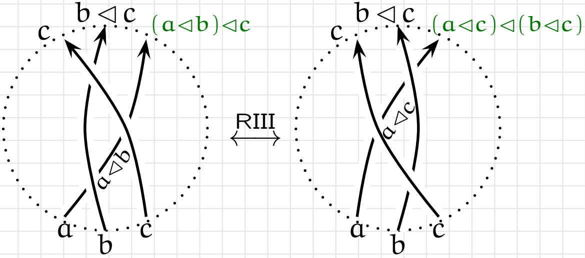
C Self-distributive structures



colorings by $(S, \triangleleft)$

"SD braiding"

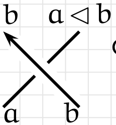
$\sigma_{SD} : (a,b) \mapsto (b, a \triangleleft b)$



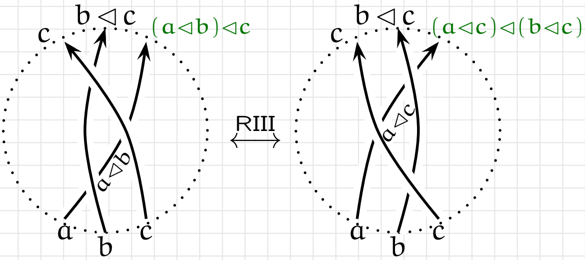
YBE $\longleftrightarrow$ RIII $\longleftrightarrow$ $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$ (SD)

3 Alg. structures via braidings

(C) Self-distributive structures



colorings by $(S, \triangleleft)$



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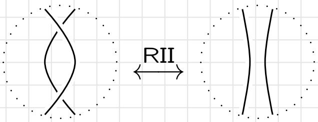
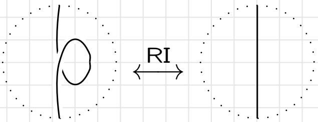
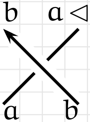
YBE	$\longleftrightarrow$	RIII	$\longleftrightarrow$	$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$	(SD)
$\exists \sigma_{SD}^{-1}$	$\longleftrightarrow$	RII	$\longleftrightarrow$	$a \mapsto a \triangleleft b$ bijective	(Inv)
		RI	$\longleftrightarrow$	$a \triangleleft a = a$	(Idem)





3 Alg. structures via braidings

Ⓒ Self-distributive structures



colorings by $(S, \lhd)$

"SD braiding"

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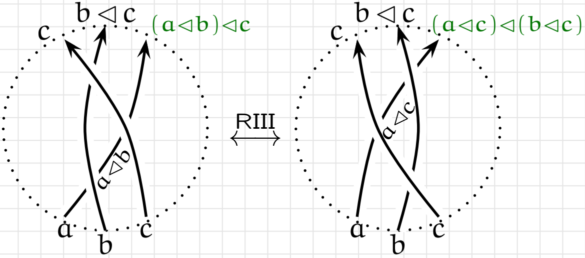

YBE	$\longleftrightarrow$	RIII	$\longleftrightarrow$	$(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$	(SD)
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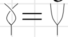
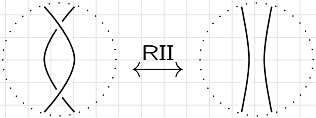
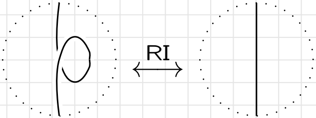
	$\longleftrightarrow$	RI	$\longleftrightarrow$	$a \lhd a = a$	(Idem)

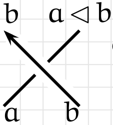




$\Delta_{SD} : a \mapsto (a, a)$

3 Alg. structures via braidings

Ⓒ Self-distributive structures

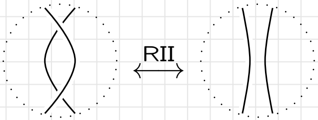


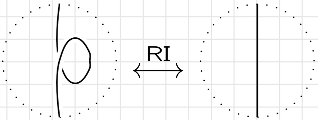
colorings by $(S, \triangleleft)$

Joyce, Matveev 1982:

knot invariants $\overset{\text{colorings}}{\leftsquigarrow}$ quandle

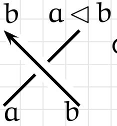
pos. braids	RIII	$\longleftrightarrow$	$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$	shelf
braids	RII	$\longleftrightarrow$	$a \mapsto a \triangleleft b$ bijective	rack
knots	RI	$\longleftrightarrow$	$a \triangleleft a = a$	quandle





3 Alg. structures via braidings

Ⓒ Self-distributive structures



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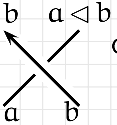
Ex.: → Conjugation quandles: (group G, $g \triangleleft h = h^{-1}gh$)

coloring rule $\longleftrightarrow$ Wirtinger presentation rule,

colorings $\longleftrightarrow$ $\mathrm{Rep}(\pi_1(\mathbb{R}^3 \backslash K), G)$.

3 Alg. structures via braidings

Ⓒ Self-distributive structures



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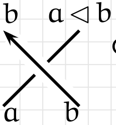
colorings $\longleftrightarrow$ n-Fox colorings.

3 Alg. structures via braidings

Ⓒ Self-distributive structures



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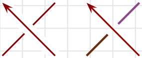
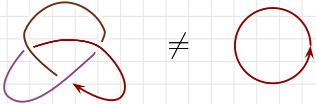
$n = 3$





Ⓒ Self-distributive structures

diagrams:	D	$\overset{\text{R-move}}{\rightsquigarrow}$	D'
colorings:	$\mathcal{C}$	$\rightsquigarrow$	$\mathcal{C}'$
coloring sets:	$\mathcal{C}ol_S(\mathrm{D})$	$\overset{\text{bij.}}{\longleftrightarrow}$	$\mathcal{C}ol_S(\mathrm{D}')$
counting invariants:	$\#\mathcal{C}ol_S(\mathrm{D})$	$=$	$\#\mathcal{C}ol_S(\mathrm{D}')$

③ Alg. structures via braidings

Ⓒ Self-distributive structures

diagrams:	D	$\overset{\text{R-move}}{\rightsquigarrow}$	D'
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Question: Extract more information?

Idea: Some "weight" ω s.t. $\omega(\mathcal{C}) = \omega(\mathcal{C}')$

$\Longrightarrow \quad \{\omega(\mathcal{C}) \mid \mathcal{C} \in \mathcal{C}ol_S(D)\} = \{\omega(\mathcal{C}') \mid \mathcal{C}' \in \mathcal{C}ol_S(D')\}.$

3 Alg. structures via braidings

Ⓒ Self-distributive structures

diagrams:	D	$\overset{\text{R-move}}{\rightsquigarrow}$	D'
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<u>Answer</u>: <u>quandle cocycle invariants</u> (*Carter-Jelsovsky-Kamada-Langford-Saito 1999*).

$\phi : S \times S \to A \;\rightsquigarrow$
<u>Boltzmann weight</u>:

$$\omega_\phi(\mathcal{C}) = \sum \pm\phi(a, b)$$

C Self-distributive structures

Rack & quandle cohomology theories

(*Fenn-Rourke-Sanderson 1995, Carter et al. 1999*)

Motivation:

→ $\{\omega_\phi(\mathcal{C}) \mid \mathcal{C} \in \mathcal{Col}_S(D)\}$ yields a braid / knot invariant when ϕ is a rack / quandle 2-cocycle;

→ this invariant is trivial when ϕ is a coboundary;

3 Alg. structures via braidings

Ⓒ Self-distributive structures

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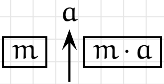
→ adding coefficients allows to color diagram regions:

rack module $M \ni$ 

$$\omega_\phi(\mathcal{C}) = \sum \pm\phi(m, a, b)$$

→ everything generalizes to $K^{n-1} \hookrightarrow \mathbb{R}^{n+1}$.

3 Alg. structures via braidings

Ⓒ Self-distributive structures

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rack module $\quad M \ni \boxed{m} \overset{a}{\uparrow} \boxed{m \cdot a}$

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Inspiration: (Co-)homology theories for associativity.

Question (Przytycki): Explain the parallels between the associative and the SD worlds?

3 Alg. structures via braidings

Ⓒ Self-distributive structures

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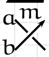

→ everything generalizes to $K^{n-1} \hookrightarrow \mathbb{R}^{n+1}$.

Inspiration: (Co-)homology theories for associativity.

Question (Przytycki): Explain the parallels between the associative and the SD worlds?

Answer: Common braided interpretation.

3 Alg. structures via braidings

Ⓒ Self-distributive structures

Shelf $(S, \triangleleft) \rightsquigarrow$ $\sigma_{SD} \colon (a, b) \mapsto (b, a \triangleleft b)$

✓ YBE for σ_{SD} $\iff$ SD for $\triangleleft$

✓ A fully faithful functor $\mathbf{Shelf} \hookrightarrow \mathbf{Br}$.

✓ σ_{SD} is invertible $\iff$ $(S, \triangleleft)$ is a rack.

✓ Braided modules for $(V, \sigma_{SD}) \longleftrightarrow$ rack modules for $(S, \triangleleft)$.

✓ $\Delta_{SD} \colon a \mapsto (a, a)$ $\rightsquigarrow$ weak braided coalgebra if $(S, \triangleleft)$ is a quandle.

✓ Braided homologies for (V, σ_{SD}) include rack, quandle, and other SD homologies.

4 Multi-component braidings

Question: How to treat more complicated structures?

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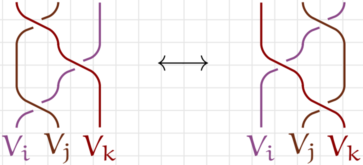
Braided system: $V_1, V_2, \ldots, V_r$ and $\sigma^{i,j} : V_i \otimes V_j \to V_j \otimes V_i$, $i \leqslant j$, satisfying the colored Yang-Baxter equation (cYBE):

$$\sigma_1^{j,k} \circ \sigma_2^{i,k} \circ \sigma_1^{i,j} = \sigma_2^{i,j} \circ \sigma_1^{i,k} \circ \sigma_2^{j,k}$$

$V_i \otimes V_j \otimes V_k \to V_k \otimes V_j \otimes V_i, i \leqslant j \leqslant k$



The collection $(\sigma^{i,j})$ satisfying cYBE is a multi-braiding.

4 Multi-component braidings

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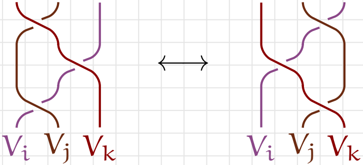
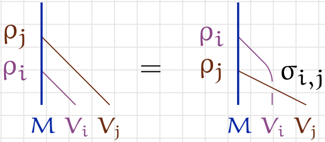


The collection $(\sigma^{i,j})$ satisfying cYBE is a multi-braiding.

Left braided V-module:
$(M, (\rho_i : M \otimes V_i \to M))$ s.t.



$i \leqslant j$

4 Multi-component braidings

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Braided system: $V_1, V_2, \ldots, V_r$ and $\sigma^{i,j} : V_i \otimes V_j \to V_j \otimes V_i$, $i \leqslant j$, satisfying the colored Yang-Baxter equation (cYBE):

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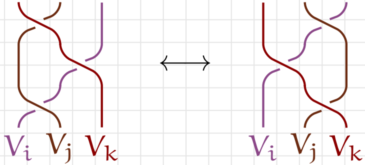

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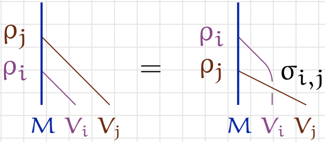


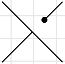
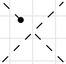
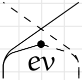
$i \leqslant j$

Theorem (L., 2013): $M \otimes T(V_1) \otimes \cdots \otimes T(V_r) \otimes N$ carries a family of differentials $\delta^{(\alpha,\beta)} = \alpha^{\bullet}\delta + \beta\,\delta^{\bullet}$, $\alpha, \beta \in \mathbb{k}$.

4 Multi-component braidings

Finite-dim. bialgebra $H \rightsquigarrow$

$$(H, H^*; \sigma_{H,H} = \sigma^r_{Ass}(H), \sigma_{H^*,H^*} = \sigma_{Ass}(H^*), \sigma_{H,H^*} = \sigma_{YD})$$

$\sigma_{H,H} =$  $\sigma_{H^*,H^*} =$  $\sigma_{H,H^*} =$ 

$$h \otimes l \mapsto \langle l_{(1)}, h_{(2)} \rangle \, l_{(2)} \otimes h_{(1)}$$

✓ YBE on $H \otimes H^* \otimes H^*$ $\underset{\text{(un. ax.)}}{\Longleftrightarrow}$ bialgebra compatibility

4 Multi-component braidings

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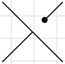
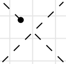
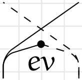
$\sigma_{H,H} =$  $\sigma_{H^*,H^*} =$  $\sigma_{H,H^*} =$ 

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✓ YBE on $H \otimes H^* \otimes H^*$ $\underset{\text{(un. ax.)}}{\Longleftrightarrow}$ bialgebra compatibility

✓ A fully faithful functor ${}^*\mathbf{Bialg} \hookrightarrow {}^*_2\mathbf{BrSyst}^{\bullet}_{\bullet}$.

4 Multi-component braidings

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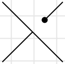
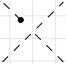
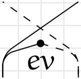
$\sigma_{H,H} =$  $\sigma_{H^*,H^*} =$  $\sigma_{H,H^*} =$ 

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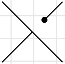
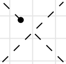
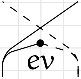
$\sigma_{H,H} =$  $\sigma_{H^*,H^*} =$  $\sigma_{H,H^*} =$ 

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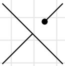
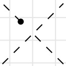
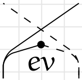
$\sigma_{H,H} =$  $\sigma_{H^*,H^*} =$  $\sigma_{H,H^*} =$ 

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→ Hopf bimodules are modules over the Heisenberg double

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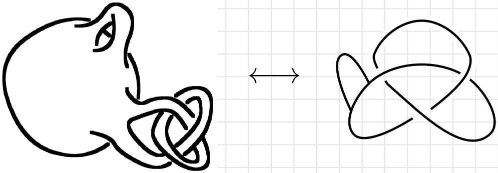
knotted handle-bodies



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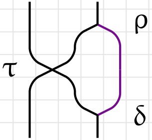

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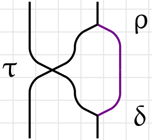

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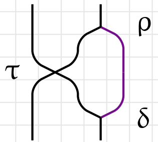

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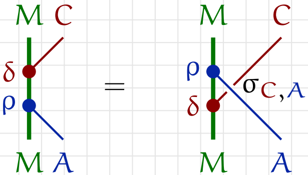

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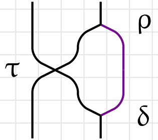

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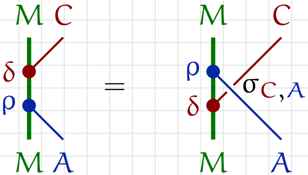

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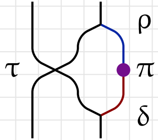

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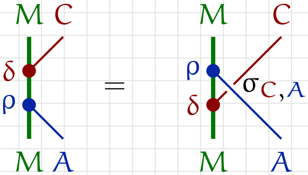

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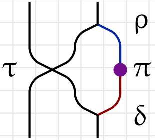

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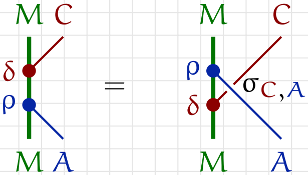

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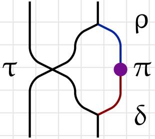

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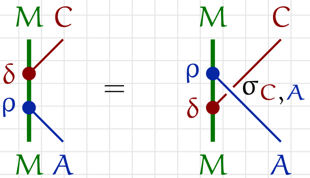

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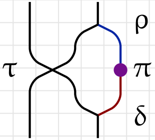

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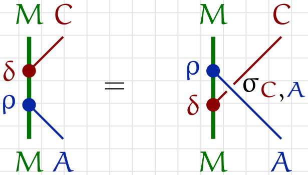

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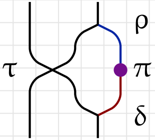

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