# On left and right nilpotency for skew left braces: a Jodan-Hölder like theorem

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## Preliminaries and notation

**Definition.** A *skew left brace* $(B,+,\cdot)$ is a set $B$ endowed with two group structures $(B,+)$ and $(B,\cdot)$ satisfying the following property:

$$a\cdot(b+c) = a\cdot b - a + a\cdot c, \quad \text{for every } a,b,c \in B$$

Throughout this poster $B$ denotes a finite skew left brace

**Definition.** Let $\mathfrak{X}$ be a class of groups. If $(B,+)$ belongs to $\mathfrak{X}$, then $B$ is called a *skew left brace of $\mathfrak{X}$-type*.

*Rump's braces are skew left braces of abelian type.*

**Notation.** The map $\lambda\colon (B,\cdot) \to \operatorname{Aut}(B,+)$ denotes the homomorphism $a \mapsto \lambda_a$, defined as $\lambda_a(b) := -a + ab$, for every $a,b \in B$.
We will consider the *star operation* $*\colon B\times B \to B$, defined as $a*b = \lambda_a(b) - b$, for every $a,b\in B$. If $X$ and $Y$ are subsets $B$,

$$X * Y := \langle x*y : x\in X, y\in Y\rangle_+$$

## Ideals and series

**Definition.** A set $\emptyset \neq I$ of $B$ is called a *left ideal* if $I$ is a subgroup of $(B,+)$ such that $\lambda_a(I)\subseteq I$, for all $a\in B$. A left ideal $I$ is said to be an *ideal* if $I$ is normal in $(B,+)$ and $I*B\subseteq I$.

**Proposition 1** ([CSV19, Proposition 1.6], [GV17, Lemma 2.5] and ). *The following hold:*

1. $\operatorname{Fix}(B) = \{x\in B : \lambda_a(x) = x$, *for all* $a\in B\}$ *is a left ideal of* $B$.
2. $\operatorname{Soc}(B) = \{x \in B : a + x = x + a \wedge \lambda_x(a) = a$, *for all* $a\in B\} = \ker(\lambda)\cap Z(B,+)$ *is an ideal of* $B$.

Following [Rum07] we define

$$B^1 = B \qquad B^{(1)} = B$$
$$B^{n+1} = B * B^n \qquad B^{(n+1)} = B^{(n)} * B, \text{ for every } n\geq 1$$

Each $B^i$ and $B^{(i)}$ are, respectively, a left ideal and an ideal of $B$. The series $B^1 \supseteq \ldots \supseteq B^n \supseteq \ldots$ and $B^{(1)} \supseteq \ldots \supseteq B^{(n)} \supseteq \ldots$ are called, respectively, the *left and right series* of $B$.

**Definition.** $B$ is said to be *left nilpotent* if $B^m = 0$, for some $m\geq 1$. Analogously, $B$ is said to be *right nilpotent* if $B^{(m)} = 0$, for some $m \geq 1$.

**Definition.** We say that a sequence of ideals of $B$

$$0 = I_0 \subseteq I_1 \subseteq \ldots \subseteq I_n = B$$

is

- a *chief series* if $I_{i+1}/I_i$ is a minimal ideal of $B/I_i$, for all $0\leq i<n$.
- an *s-series* if $I_{i+1}/I_i \subseteq \operatorname{Soc}(B/I_i)$, for all $0\leq i < n$.
- an *f-series* if $I_{i+1}/I_i \subseteq \operatorname{Fix}(B/I_i)$, for all $0 \leq i < n$.

Each factor $I_{i+1}/I_i$ is called, respectively, a *chief factor*, an *s-factor* or an *f-factor*.

**Definition.** We call a sequence of left ideals $0 = L_0 \subseteq L_1 \subseteq \ldots \subseteq L_n = B$ an *f-series* if, $B * L_{i+1} \subseteq L_i$, for all $0\leq i<n$.

## Results

Our first main result is a general version of a Jordan-Hölder theorem for skew left braces:

**Theorem 1.** *For any two chief series of* $B$,

$$0 = I_0 \subseteq I_1 \subseteq \ldots \subseteq I_n = B$$
$$0 = J_0 \subseteq J_1 \subseteq \ldots \subseteq J_m = B$$

*it holds that* $n = m$ *and there exists a permutation* $\sigma \in \operatorname{Sym}(n)$ *such that* $I_{i+1}/I_i$ *is isomorphic to* $J_{\sigma(i+1)}/J_{\sigma(i)}$ *and* $I_{i+1}/I_i$ *is an s-factor (f-factor) if, and only if,* $J_{\sigma(i+1)}/J_{\sigma(i)}$ *is an s-factor (respectively, f-factor).*

The proof is based on these two key lemmas. The first one is the analogous one for skew left braces in [FC21]

**Lemma 1.** *Assume that* $I$, $J$ *and* $L$ *are ideals of* $B$ *with* $L\subseteq I$. *Then,*

$$\frac{IJ}{LJ} = \frac{ILJ}{LJ} \cong \frac{I}{LJ\cap I} = \frac{I}{L(J\cap I)}$$

and the second one is the following

**Lemma 2.** *Let* $I$ *and* $J$ *two distinct minimal ideals of* $B$ *and set* $L = IJ$. *Then,*

1. $I\in\operatorname{Soc}(B)$ *if, and only if,* $L/J \in \operatorname{Soc}(B/J)$
2. $J\in\operatorname{Soc}(B)$ *if, and only if,* $L/I\in\operatorname{Soc}(B/I)$

Our second main result is a characterisation of right nilpotency for skew left braces of nilpotent-type.

**Theorem 2.** *Suppose that* $B$ *is of nilpotent-type. Then,* $B$ *is right nilpotent if, and only if, every chief factor is an s-factor.*

For the proof of our second main result we need a lemma which can be found in [CSV19]

**Lemma 3.** *Suppose that* $B$ *is of nilpotent-type and let* $I$ *be a minimal of* $B$. *Then,* $I\subseteq \operatorname{Soc}(B)$.

As a corollary of Theorem 2 we obtain a characterisation of right nilpotency on skew left braces of nilpotent-type

**Corollary 1.** *Suppose that* $B$ *is of nilpotent-type. Then,* $B$ *is right nilpotent if, and only if, it admits an s-series.*

Finally, we prove a characterisation of left nilpotency in terms of f-series which does not require $B$ to be of nilpotent-type.

**Proposition 2.**

*$B$ is left nilpotent if, and only if, it admits an f-series of left ideals.*

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