Monoids of left non-degenerate solutions of YBE

Algebra Days in Caen

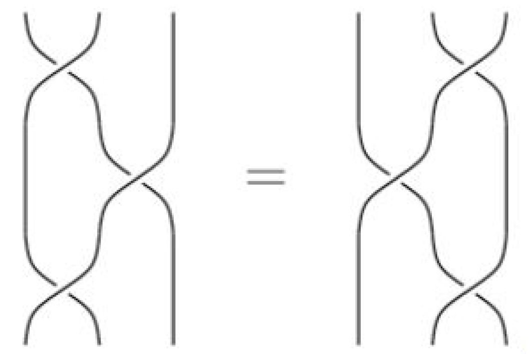
General notations and definitions

A set-theoretic solution (X, r) is a set X and a map $X \times X \longrightarrow X \times X$ such that

$$(r \times \mathrm{id}_X)(\mathrm{id}_X \times r)(r \times \mathrm{id}_X) = (\mathrm{id}_X \times r)(r \times \mathrm{id}_X)(\mathrm{id}_X \times r).$$

Denote $r(x, y) = (\lambda_x(y), \rho_y(x))$.

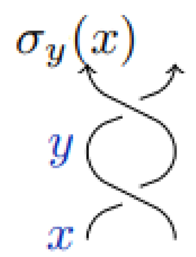
In a picture:



A set-theoretic solution is called:

- left non-degenerate, if λ_x is bijective for all $x \in X$,
- right non-degenerate, if ρ_x is bijective for all $x \in X$,
- non-degenerate, if (X, r) is both left and right non-degenerate,
- bijective (resp. involutive), if r is bijective (resp. involutive).

Derived Structure monoid



$$A(X, r) = \langle x \in X \mid xy = y\sigma_y(x) \rangle .$$

Finite non-degenerate solution (X, r)**:**[3,4]

- $A(X, r)$ is central-by finite,
- $A(X, r)$ is cancellative iff $A(X, r)$ is free abelian,
- $KA(X, r)$ is Noetherian, PI and finite GKdimension.

Finite left non-degenerate solution (X, r)**:**[2]

Let $X = \{x_1, ..., x_n\}$ and $v \in \mathbb{Z}_{>0}$ s.t. σ_x^v is idempotent for all $x \in X$.

- $A(X, r)$ is finite left module over

$$B = A(Y, s_Y) = \langle y_1 = vx_1, ..., y_n = vx_n \rangle$$

Denote $t_k = y_1 + ... + y_k$, $\kappa \in S_n$ and $B_{\kappa(t_k)} = \langle y_{\kappa(i)} \mid 1 \le i \le k \rangle$

- $B_{\kappa(t_k)} + \kappa(t_k)$ is commutative.

Denote $B_k = \cup_{\kappa \in S_n} B_{\kappa(t_k)} + \kappa(t_k)$.

- $B_n \subseteq B_n \cup B_{n-1} \subseteq ... \subseteq B$ is an ideal chain of B.

Let K be any field, each factor of the ideal chain

$$K[B_n] \subseteq K[B_n] + K[B_{n-1}] \subseteq ... \subseteq K[B_n] + ... + K[B_1] \subseteq K[B]$$

is a Noetherian left $K[B]$ module and a sum of commutative rings.

Conclusion:

$KA(X, r)$ is a left Noetherian, PI-algebra of finite GKdimension.

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Structure monoid

$$M(X, r) = \langle x \in X \mid xy = uv \text{ if } r(x, y) = (u, v) \rangle .$$

Finite non-degenerate solution (X, r)**:**[3,4]

- $M(X, r)$ is abelian-by-finite
- $M(X, r)$ is cancellative iff r is involutive
- $KM(X, r)$ is Noetherian, PI and finite GKdimension.

Finite left non-degenerate solution (X, r)**:**

- Structure of $M(X, r)$ is current joint work with Colazzo, Jespers, Kubat and Verwimp.

Challenges and approaches:

- There exists a semigroup embedding

$$\varphi : M(X, r) \longrightarrow A(X, r) \rtimes \langle \lambda_x \mid x \in X \rangle ,$$

with $\varphi(x) = (x, \lambda_x)$. Denote the projection $\pi : M(X, r) \longrightarrow A(X, r)$.

- The square map $x \mapsto \lambda_x^{-1}(x)$ may be non-bijective.
- There does not always exist a $n \in \mathbb{Z}_{>0}$ s.t.

$$\lambda_{\pi^{-1}(nx)} = \mathrm{id}_{M(X,r)}.$$

YB-semitrusses

Definition:

$(B, +, \circ, \lambda, \sigma)$ is a YB-semitruss, if, for any $a, b, c \in B$,

- $(B, +)$ and $(B, \circ)$ are semigroups with $a \circ b = a + \lambda_a(b)$,
- $\lambda_a \in \mathrm{Aut}(B, +)$ and $\lambda_a \lambda_b = \lambda_{a \circ b}$,
- $a + b = b + \sigma_b(a)$, $\sigma_a \in \mathrm{End}(B, +)$ and $\sigma_{a+b} = \sigma_b \sigma_a$,
- $\sigma_{\lambda_a(c)} \lambda_a(b) = \lambda_a \sigma_c(b)$

An example is $M(X, r)$ **for a left non-degenerate solution!**

YB-semitrusses govern left non-degenerate solutions:

- A LND solution (X, r) has an associated YB-semitruss $M(X, r)$,
- Any YB-semitruss B has an associated LND solution on B.

We classified the YB-semitrusses with left simple semigroup

$$\mathcal{C}(X, r) = \langle \sigma_x \mid x \in X \rangle .$$

Important result:

The following are equivalent for a finite LND solution (X, r):

- **the solution** (X, r) **is right non-degenerate,**
- **the solution** (X, r) **is bijective.**

Castelli, Catino and Stefanelli[1] already showed that finite bijective LND solutions are right non-degenerate.

References

[1] M. Castelli, F. Catino, P. Stefanelli, Left non-degenerate set-theoretic solutions of the Yang-Baxter equation and dynamical extensions of q-cycle sets, arXiv:2001.10774.

[2] I. Colazzo, E. Jespers, A. Van Antwerpen, C. Verwimp, Left non-degenerate set-theoretic solutions of the Yang-Baxter equation and semitrusses, arXiv:2109.04978.

[3] E. Jespers, L.Kubat, A. Van Antwerpen, The structure monoid and algebra of a non-degenerate set-theoretic solution of the Yang-Baxter equation, Trans.Amer.Math.Soc 372 (2019), no.10, 7191-7223 + Corrigendum et Addendum

[4] V. Lebed and L. Vendramin, On structure groups of set-theoretic solutions to the Yang-Baxter equation, Proc. Edinb.Math.Soc. 62 (2019), no. 3, 683-717.