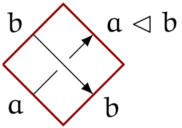
Unexpected applications of homotopical algebra to knot theory

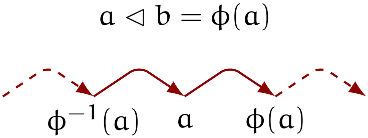
(Honest title: Homology of permutation racks)

Victoria LEBED, University of Caen Normandy (France)
Joint work with Markus SZYMIK, NTNU (Norway)

(Virtual version of) Washington, March 2021



I. From topology to algebra



II. From algebra to topology

0 Parallel session

A challenge (for those who know rack homology better than I do):

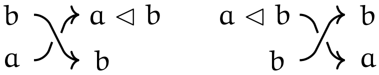
Compute the full rack homology of the permutation rack
$(S\,,\ a \triangleleft b = \phi(a))$.

1 How topologists discovered self-distributivity

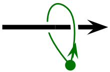
D. Joyce & S. Matveev, knot colorists separated by the Iron Curtain:

Take a set S endowed with a binary operation $\triangleleft$.

$(S, \triangleleft)$-colourings for braid diagrams:



cf. Wirtinger presentation of $\pi_1([0,1] \times \mathbb{R}^2 \setminus \beta)$:



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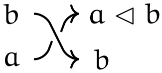

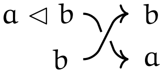

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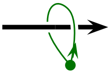
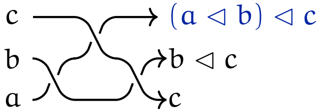
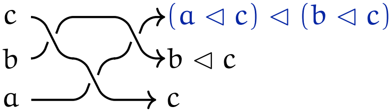




RIII
$\sim$



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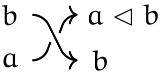
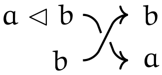
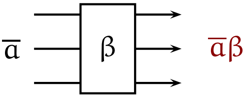




$\mathrm{End}(S^n) \leftarrow B_n^+$	RIII	$(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$	shelf
$\mathrm{Aut}(S^n) \leftarrow B_n$	& RII	$\forall b,\ a \mapsto a \lhd b$ is bijective	rack
$S \hookrightarrow (S^n)^{B_n}$	& RI	$a \lhd a = a$	quandle

$a \mapsto (a, \ldots, a)$



2 Braids and self-distributivity

$\mathrm{End}(S^n) \leftarrow B_n^+$	RIII	$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$	shelf
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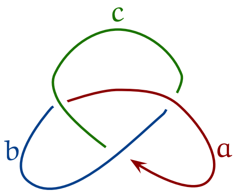
$S \hookrightarrow (S^n)^{B_n}$	& RI	$a \triangleleft a = a$	quandle

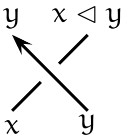
$a \mapsto (a, \dots, a)$

S	$a \triangleleft b$	$(S, \triangleleft)$ is a	in braid theory
${}_{\mathbb{Z}[t^{\pm 1}]}\mathrm{Mod}$	$ta + (1-t)b$	quandle	Burau: $B_n \to GL_n(\mathbb{Z}[t^{\pm}])$
group	$b^{-1}ab$	quandle	Artin: $B_n \hookrightarrow \mathrm{Aut}(F_n)$
twisted linear quandle			Lawrence–Krammer–Bigelow
$\mathbb{Z}$	$a+1$	rack	$\lg(w), \mathrm{lk}_{i,j}$
free shelf			Dehornoy: order on B_n

3 Knots and self-distributivity

$(S, \triangleleft)$-colourings for **knot** diagrams:





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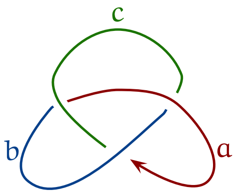

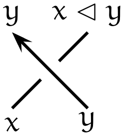

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Proposition: $(S, \triangleleft)$ is a quandle $\Longrightarrow$

$\#\{\, (S, \triangleleft)\text{-colourings of diagrams} \,\}$ is a knot invariant.

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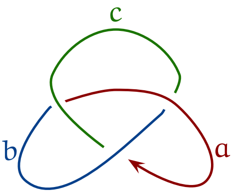

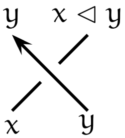

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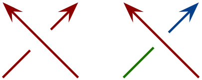
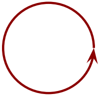
Proposition: $(S, \lhd)$ is a quandle $\implies$

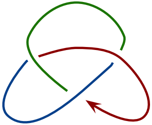
$\#\{\,(S, \lhd)\text{-colourings of diagrams}\,\}$ is a knot invariant.

Example: $(\mathbb{Z}_3,\ a \lhd b = 2b - a)$



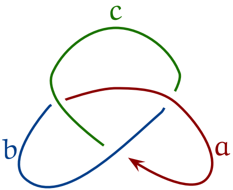


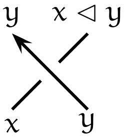
3 colourings



9 colourings

$(S, \triangleleft)$-colourings for knot diagrams:





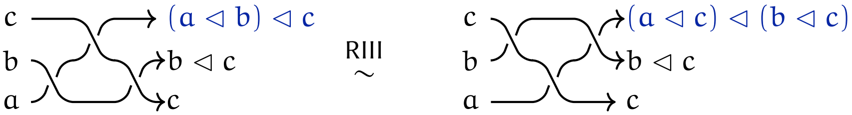
$$a \triangleleft b = c, \quad b \triangleleft c = a, \quad c \triangleleft a = b$$

Theorem (*Joyce & Matveev '82*):

- $\# \mathrm{Col}_{S,\triangleleft}(D) = \# \mathrm{Hom}_{\mathtt{Quandle}}(Q(K), S)$,
- $Q(K)$ = fundamental quandle of K

  (a weak universal knot invariant).

4 The homology comes in



diagrams:	D	$\overset{\text{R-move}}{\rightsquigarrow}$	D'
colorings:	$\mathcal{C}$	$\rightsquigarrow$	$\mathcal{C}'$
coloring sets:	$\mathrm{Col}_{S,\lhd}(D)$	$\overset{1:1}{\longleftrightarrow}$	$\mathrm{Col}_{S,\lhd}(D')$

Counting invariants: $\#\,\mathrm{Col}_{S,\lhd}(D) = \#\,\mathrm{Col}_{S,\lhd}(D')$.

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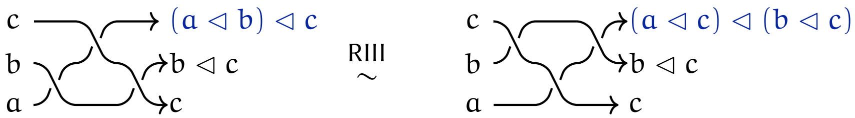

$$\begin{array}{rccc} \text{diagrams:} & D & \overset{\text{R-move}}{\rightsquigarrow} & D' \\ \text{colorings:} & \mathcal{C} & \rightsquigarrow & \mathcal{C}' \\ \text{coloring sets:} & \mathrm{Col}_{S,\triangleleft}(D) & \overset{1:1}{\longleftrightarrow} & \mathrm{Col}_{S,\triangleleft}(D') \end{array}$$

Counting invariants: $\#\,\mathrm{Col}_{S,\triangleleft}(D) = \#\,\mathrm{Col}_{S,\triangleleft}(D')$.

Question: Extract more information?

$$\omega(\mathcal{C}) = \omega(\mathcal{C}')$$

$$\Downarrow$$

$$\{\, \omega(\mathcal{C}) \,|\, \mathcal{C} \in \mathrm{Col}_{S,\triangleleft}(D) \,\} = \{\, \omega(\mathcal{C}') \,|\, \mathcal{C}' \in \mathrm{Col}_{S,\triangleleft}(D') \,\}.$$

Answer (*Carter–Jelsovsky–Kamada–Langford–Saito* '03): State-sums over crossings, and Boltzmann weights:

$$\phi\colon S \times S \to \mathbb{Z}_m \qquad \rightsquigarrow \qquad \omega_\phi(\mathcal{C}) = \sum \pm\phi(a, b)$$

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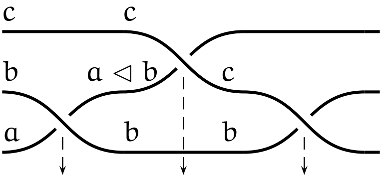
Conditions on ϕ:



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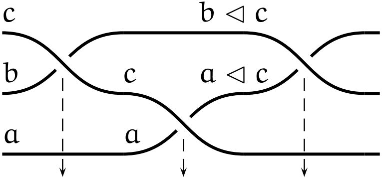
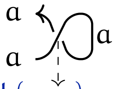
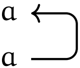


$$\phi(a, b) + \phi(a \triangleleft b, c) + \cancel{\phi(b, c)} = \cancel{\phi(b, c)} + \phi(a, c) + \phi(a \triangleleft c, b \triangleleft c)$$



RI
$\sim$



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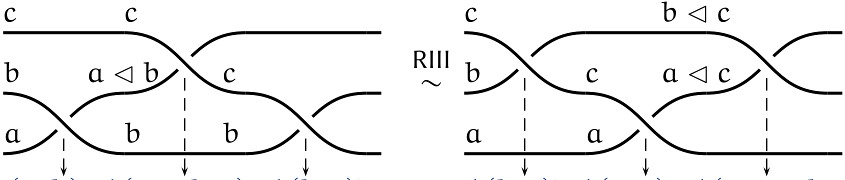
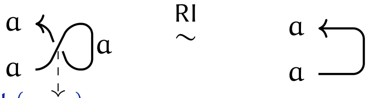


$$\phi(a,b)+\phi(a \lhd b, c)+\cancel{\phi(b,c)} \;=\; \cancel{\phi(b,c)}+\phi(a,c)+\phi(a \lhd c, b \lhd c)$$



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Quandle cocycle invariants: $\{\, \omega_\phi(\mathcal{C}) \;\big|\; \mathcal{C} \in \mathrm{Col}_{S,\lhd}(D) \,\}$.

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Quandle cocycle invariants $\supsetneq$ counting invariants.

Conjecture (*Clark–Saito–…*):

Finite quandle cocycle invariants distinguish all knots.

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Wish:

$d^{n+1}\phi = 0 \implies \phi$ refines counting invariants for n-knots,

$\phi = d^n\psi \implies$ the refinement is trivial.

5 The desired cohomology theory

Fenn et al. '95 & *Carter et al.* '03 & *Graña* '00:

Shelf $(S, \triangleleft)$ & abelian group X $\rightsquigarrow$ cochain complex

$$C^k_R(S, X) = \mathrm{Map}(S^{\times k}, X),$$

$$(d^k_R f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} (f(a_1, \ldots, \widehat{a_i}, \ldots, a_{k+1}) - f(a_1 \triangleleft a_i, \ldots, a_{i-1} \triangleleft a_i, a_{i+1}, \ldots, a_{k+1}))$$

$\rightsquigarrow$ Rack cohomology $H^k_R(S, X) = \mathrm{Ker}\, d^k_R / \mathrm{Im}\, d^{k-1}_R$.

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Quandle $(S, \triangleleft)$ & abelian group $X \rightsquigarrow$ sub-complex of (C^k_R, d^k_R):

$$C^k_Q(S, X) = \{ f \colon S^{\times k} \to X \mid f(\ldots, a, a, \ldots) = 0 \}$$

$\rightsquigarrow$ Quandle cohomology $H^k_Q(S, X)$.

This is what we were looking for! This construction yields:

- ✓ Boltzmann weights for constructing higher knot invariants (powerfull and easy to compute);
- ✓ an important class of braided vector spaces giving nice Hopf algebras;
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Problem: Full rack/quandle (co)homology of a rack is hard to compute.

We will give a partial overview of available tools.

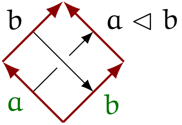
6 Topological realization

Fenn–Rourke–Sanderson '95:

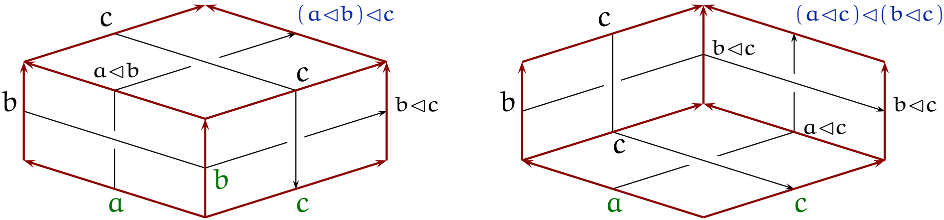
Shelf $(S, \lhd)$ $\rightsquigarrow$ rack (= classifying) space $B(S)$. It is a CW-complex:

deg 0: $*$ deg 1: $* \xrightarrow{a} *$

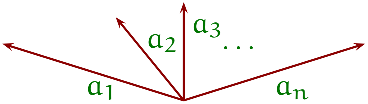
deg 2:



deg 3:



deg n: $\coprod_{S^{\times n}} [0,1]^n$



The coloring continues uniquely to other edges of $[0,1]^n$.

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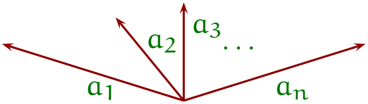

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Boundaries: usual topological ones.

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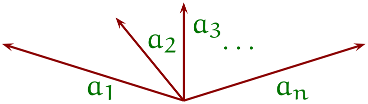

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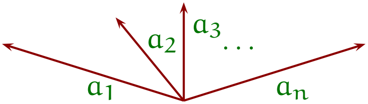

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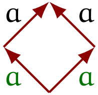
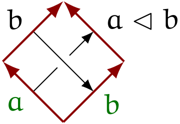
Nosaka '11: To get quandle cohomology, add 3-dimensional cells bounding



Proposition: $\boxed{\pi_1(B(S)) \cong \mathrm{As}(S)}$,

where $\mathrm{As}(S) := \langle S \mid a\,b \;=\; b\,(a \lhd b)\rangle$ is the associated group of $(S, \lhd)$.



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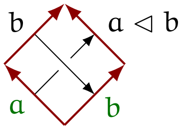

Computations (*Fenn–Rourke–Sanderson* '07):

1) Trivial quandle $T_n = (\{1, \ldots, n\},\; a \lhd b = a)$: $\quad B(T_n) \cong \Omega(\vee_n \mathbb{S}^2)$.

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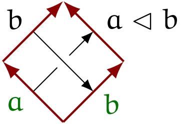

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2) Free rack on n generators FR_n: $\quad B(FR_n) \cong \vee_n \mathbb{S}^1$.

7 Rack cohomology vs group cohomology

The associated group of $(S, \triangleleft)$:

$$\mathrm{As}(S) := \langle S \mid a\,b \;=\; b\,(a \triangleleft b)\rangle$$

Theorem (*Joyce* '82): One has a pair of adjoint functors

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Theorem (*Fenn–Rourke–Sanderson* '95): There is a graded algebra morphism $HH^\bullet(\mathrm{As}(S), X) \to H^\bullet_R(S, X)$.

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It works, and yields interesting invariants!

Homotopical tools: framework

Theorem (*Szymik* '19): Quandle cohomology is a Quillen cohomology.

Applications:

- ✓ excision isomorphisms;
- ✓ Mayer–Vietoris exact sequences.

Homotopical tools: example

A permutation ϕ on a set S $\rightsquigarrow$ permutation rack $(S\,,\ a \triangleleft_{\phi} b = \phi(a))$.

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✓ $\beta_0 = 1,\ \beta_1 = r,\ \beta_{n+2} = (r-1)\beta_{n+1} + r_f \beta_n, \qquad n \geqslant 0$;

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Trick: $H^R_k = \mathrm{Ker}\, d^R_k / \mathrm{Im}\, d^R_{k+1}$
study chains up to boundaries, then restrict to cycles
(usually: determine cycles, then mod out the boundaries).

Step 2 Choose a simplicial resolution by free permutations $F_\bullet \to S$

$\rightsquigarrow$ a double complex $E^0_{p,q} = C^R_q(F_p)$

$\rightsquigarrow$ two spectral sequences with the same target.

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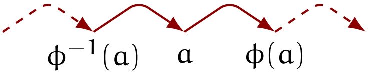

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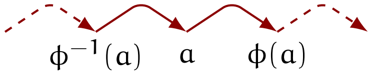

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Step 4 For the 2nd SS, show that $E^\infty = E^2$.

For this, find enough independent elements in $H^R_q(S)$.