Algebraic structures → Braidings → Homology

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§1. Introduction

general framework

algebraic structure

- monoidal category $\mathcal{C}$
- an object V
- structural morphisms $\varphi_i : V^{\otimes l_i} \to V^{\otimes r_i}$, $i \in I$
- structural relations $(R_j)_{j \in J}$

guess, §3

braiding

$\sigma : V \otimes V \to V \otimes V$

s.t. $\sigma_1 \circ \sigma_2 \circ \sigma_1 = \sigma_2 \circ \sigma_1 \circ \sigma_2 : V^{\otimes 3} \to V^{\otimes 3}$ (YB)

⚠ σ is not necessarily invertible

general recipe, §2

differential complex

$d_n : V^{\otimes n} \to V^{\otimes (n-1)}$

s.t. $d_{n-1} \circ d_n = 0 \ \forall n \geq 2$

($\mathcal{C}$ preadditive)

example

unitary associative algebra

- $\mu : V \otimes V \to V$
- (Ass)

$\nu : I \to V$

(Un)

⇕

(YB)

σ_{Ass}

bar complex

$(d_B)_n = \sum_{i=1}^{n-1} (-1)^{i-1} \mu_i$

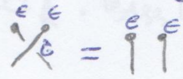
$\mu_i = Id_V^{\otimes (i-1)} \otimes \mu \otimes Id_V^{\otimes (n-i-1)}$

in $Vect_k$: $d(v_1 \otimes v_2 \otimes \dots \otimes v_n) = \mu(v_1 \otimes v_2) \otimes v_3 \otimes \dots \otimes v_n - v_1 \otimes \mu(v_2 \otimes v_3) \otimes v_4 \otimes \dots \otimes v_n + \dots$

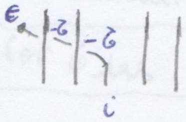
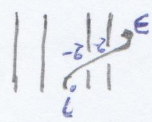
§2. Homology of braided objects in $\mathcal{C}$

$\mathcal{C}$: preadditive monoidal category

(V,σ): (weakly) braided object $\rightsquigarrow$ $(V,-\sigma)$: (weakly) braided object

$V \xrightarrow{\epsilon} I$: a cut, i.e. compatible with σ: 

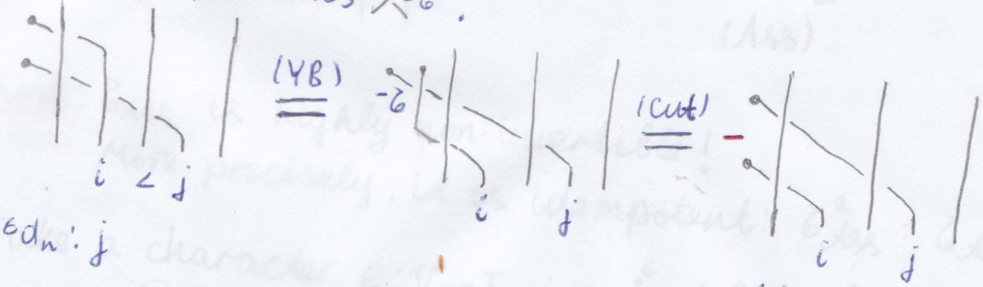
Th.: A differential bicomplex can be defined for (V,σ,ϵ) by

${}^{\epsilon}d_n = \sum_{i=1}^{n}$  $\quad d^{\epsilon}_n = (-1)^{n-1}\sum_{i=1}^{n}$ 

i.e. ${}^{\epsilon}d_{n-1} \circ {}^{\epsilon}d_n = d^{\epsilon}_{n-1} \circ d^{\epsilon}_n = {}^{\epsilon}d_{n-1} \circ d^{\epsilon}_n + d^{\epsilon}_{n-1} \circ {}^{\epsilon}d_n = 0.$

Cor: Linear combinations of ${}^{\epsilon}d$ and d^{ϵ} are differentials for V.
e.g. ${}^{\epsilon}d - d^{\epsilon}$

◇ Here $\diagdown$ denotes $\diagdown_{-\sigma}$.



${}^{\epsilon}d_n$: j

${}^{\epsilon}d_{n-1}$: i

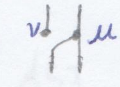
${}^{\epsilon}d_n$: i

${}^{\epsilon}d_{n-1}$: j

cancellation ◇

§3. Examples of structural braidings + §4.

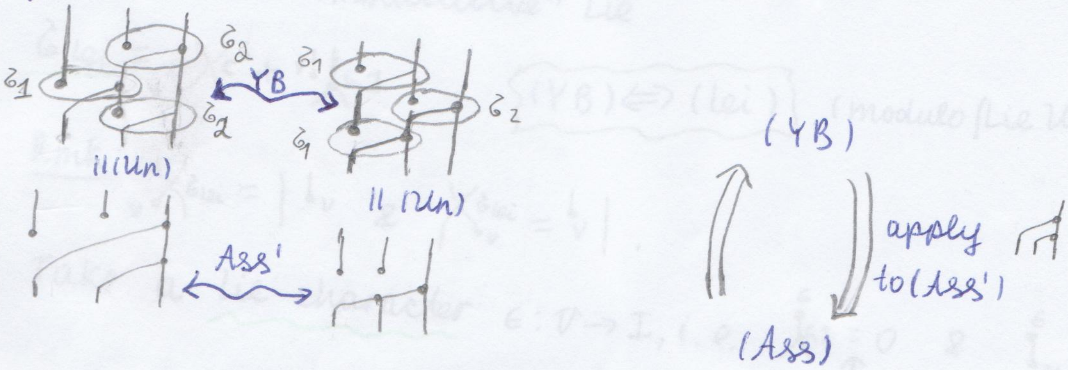
① Unitary associative algebras.

$\sigma_{Ass} = \nu \otimes \mu$  In Vect_k : $\sigma_{Ass}(v \otimes w) = 1 \otimes v \cdot w$.

Prop.: Take $V \in \mathrm{Ob}(\underset{\text{mon. cat.}}{\mathcal{C}})$, $\mu : V \otimes V \to V$, $\nu : I \to V$. Suppose that ν is a unit. Then

(YB) for σ_{Ass} $\Leftrightarrow$ (Ass) for μ

◇



◇

Rmk: σ_{Ass} is highly non-invertible!
More precisely, it is idempotent: $\sigma_{Ass}^2 = \sigma_{Ass}$.

Take a character $\epsilon : V \to I$, i.e. $\epsilon \mu = \epsilon \otimes \epsilon$ & $\epsilon \nu = Id_I$. (Char)

Lemma: character $\Leftrightarrow$ cut.

◇ $\overset{(Char)}{=}$ $\epsilon \ \epsilon$ ◇

Let $\mathcal{C}$ be preadditive monoidal. Our thm is then applicable:

$$ {}^{\epsilon}d_n = \sum_i \ldots = \sum_i (-1)^{i-1} \ldots = \underbrace{\sum_{i=2}^{n} (-1)^{i-1} \ldots}_{Bar} + \ldots $$

⇓

Prop.: (a form of) d_B is a particular case of ${}^{\epsilon}d$.

⓪ Empty structure

$(\mathcal{C}, c_{V,W})$: symmetric preadditive category

V, $\sigma_\emptyset = c_{V,V}$

Any $\epsilon : V \to I$ is a cut: $\overset{(Nat)}{=}$ $\epsilon \ \epsilon$

$$ {}^{\epsilon}d_n = \sum_{i=1}^{n} (-1)^{i-1} \, ||\underset{i}{\epsilon}|| = (d_{Koszul})_n $$

In Vect_k, $d_{Koszul}(v_1 \otimes \ldots \otimes v_n) = \sum_{i=1}^{n} (-1)^{i-1} \epsilon(v_i)\, v_1 \otimes \ldots \otimes \widehat{v_i} \otimes \ldots \otimes v_n$.

② Unitary Leibniz algebras

- $(\mathcal{C}, c_{V,W})$: symmetric preadditive category.
- $V \in \mathrm{Ob}(\mathcal{C})$
- $[,]: V \otimes V \to V$ $\quad \curlywedge_{[,]}$ $\qquad \nu: I \to V$
- $\curlywedge = \curlywedge + \curlywedge_{c_{V,V}}$ (Lei) $\qquad \nu\!\curlywedge^{[,]} = 0 = \curlywedge^{[,]}_{\nu}$ (Lie Un)

In Vect_k: $[[u,v],w] = [u,[v,w]] + [[u,w],v]$ (a form of Jacobi relation)

Leibniz = "non-commutative" Lie

$\sigma_{Lei} = \times_c + \nu \curlywedge_{[,]}$ $\qquad \boxed{(YB) \Leftrightarrow (Lei)}$ (modulo (Lie Un))

Rmk: $\nu \times^{\sigma_{Lei}} = | \, {}_\nu$ & $\times^{\sigma_{Lei}}_{\nu} = {}_\nu \, |$.

Take a Lie character $\epsilon: V \to I$, i.e. $\overset{\epsilon}{\curlywedge}_{[,]} = 0$ & $\overset{\epsilon}{|}_{\nu} = \mathrm{Id}_I$.

(in Vect_k: $\epsilon([v,w]) = [\epsilon(v), \epsilon(w)] = 0$ ∈ k is commutative)

Lemma: Lie character ⟺ cut.

$${}^{\epsilon}d_n = \sum_i \overset{\epsilon}{|}\,|\,\sigma_{Lei}\cdots\sigma_{Lei}\,|_{i} = \underbrace{\sum_{j<i} (-1)^{i-1} \,|\,|\,[,]_{j\,i}\,|}\; + \sum_i (-1)^{i-1} \,|\,|\overset{\epsilon}{|}_i|$$

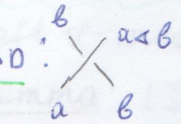
Leibniz (= "non-comm. Chevalley-Eilenberg")

In Vect_k: $d_{Lei}(v_1 \otimes \ldots \otimes v_n) = \sum_{1 \le j < i \le n} (-1)^{i-1} \, v_1 \otimes \ldots \otimes v_{j-1} \otimes [v_j, v_i] \otimes v_{j+1} \otimes \ldots \otimes \hat{v_i} \otimes \ldots \otimes v_n.$

③ Self-distributive structures

- $\mathcal{C} = Set$
- <u>shelf</u>: S, $\bullet\ S \times S \xrightarrow{\triangleleft} S$
  - $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c) \quad \forall a, b, c \in S$ (SD)

σ_{SD}:



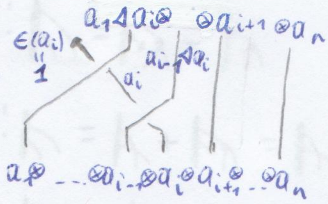
$\boxed{(YB) \Leftrightarrow (SD)}$

$\exists!\ S \xrightarrow{\epsilon} \underset{(final)}{I}$, automatically a cut.

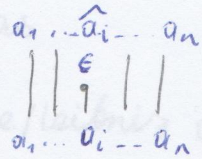
$Set \xrightarrow{Lin} Vect_k$ → preadditive

$S \mapsto kS$

$${}^{\epsilon}d_n = \sum_{i=1}^{n} (-1)^{i-1}$$



$$d^{\epsilon}_n = \sum_{i=1}^{n} (-1)^{i-1}$$



${}^{\epsilon}d - d^{\epsilon}$ defines the shelf homology.

<u>Other examples</u>: Hopf algebras, Yetter-Drinfeld modules etc.

§4. Advantages of using braidings to construct homologies

→ automatic sign treatment

→ generalized form

→ a collection of differentials (2 for each cut), often compatible

→ intuition for defining homologies

→ released conditions on structures

→ Loday's hyperboloid

ex. for associative algebras:

- $d_{n,1} := (d_B)_n$

  $d_{n,2} := \sum_{i \leq j} (-1)^{i+j} \mu_j \circ \mu_i$

  $\vdots$

  $d_{n,K}$

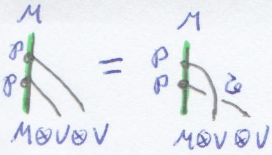
- $d_{n-K_1, K_2} \circ d_{n, K_1} = \binom{K_1 + K_2}{K_1}_{-1} d_{n, K_1 + K_2}$

↛ no information on homologies

§5. Modules over braided objects

$\mathcal{C}$: mon. category,

(V, σ): braided object.

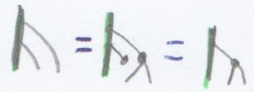
Right V-module $\overset{\text{def}}{:}$ $(M \in Ob(\mathcal{C}),\ M \otimes V \xrightarrow{\rho} M)$ s.t.  (Mod)

Left V-module: ...

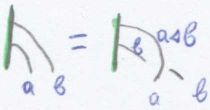
Lemma: $(I, \underset{\text{cut}}{\epsilon})$ is a V-bimodule

Analogue of our thm: bidifferential on $\underset{\text{right}}{M} \otimes V^{\otimes n} \otimes \underset{\text{left}}{N}$. } V-modules

Ex.: ⓪ $\sigma_\phi = \pm c_{V,V}$: (anti)commutativity

① σ_{Ass}:  → modules over ass. algebras

② σ_{Lei}:  → modules over Lie/Leibniz algebras

③ σ_{SD}:  → shelf modules

§6. Quantum shuffles

$\mathcal{C}$: preadd. mon. category.

(V, σ): braided object.

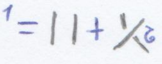
$s \in S_n \rightsquigarrow$ a shortest form $s = s_{i_1} \cdots s_{i_k} \rightsquigarrow T_s^\sigma := \sigma_{i_1} \circ \cdots \circ \sigma_{i_k} \in End_{\mathcal{C}}(V^{\otimes n})$.

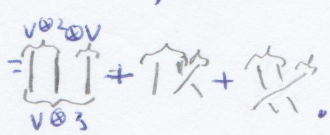
* T_s^σ is well defined
* not an action: $T_s^\sigma \circ T_{s'}^\sigma \neq T_{s \circ s'}^\sigma$ in general $\rightsquigarrow$ "quasi-action"

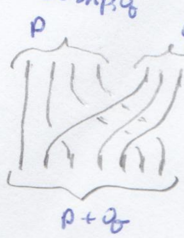
$Sh_{p,q} := \{ s \in S_{p+q} \mid s(1) < \cdots < s(p),\ s(p+1) < \cdots < s(p+q) \}$ $\overset{\text{def}}{\cdot}$ shuffles

Quantum shuffle → product: $\underset{\sigma}{Ш}$: → coproduct: $\underset{\sigma}{\overline{Ш}}^{p,q}$: $V^{\otimes p} \otimes V^{\otimes p} = V^{\otimes(p+q)} \underset{\underset{s \in Sh_{p,q}}{\sum} T^\sigma_{s^{-1}}}{\overset{\underset{s \in Sh_{p,q}}{\sum} T_s^\sigma}{\rightleftarrows}} V^{\otimes(p+q)}$ } → ass. → coass.

+ (de-)concatenation = Hopf algebra

Ex.: $\underset{\sigma}{\overline{Ш}}^{1,1}$ = ,

$\underset{\sigma}{\overline{Ш}}^{2,1}$ = .



Lemma: * (cut) $\Leftrightarrow$ $(\epsilon \otimes \epsilon) \circ \underset{-\sigma}{\overline{Ш}}^{1,1} = 0$;

* ${}^\epsilon d_n = \epsilon_1 \circ \underset{-\sigma}{\overline{Ш}}^{1,n-1}$.

Quantum shuffle proof of our theorem:

$${}^\epsilon d_{n-1} \circ {}^\epsilon d_n = (\epsilon \otimes \epsilon)_1 \circ (Id \otimes \underset{-\sigma}{\overline{Ш}}^{1,n-2}) \circ \underset{-\sigma}{\overline{Ш}}^{1,n-1} \overset{(coass)}{=} \overset{\overset{0}{\shortparallel}\,(\text{cut})}{\left((\epsilon \otimes \epsilon) \circ \underset{-\sigma}{\overline{Ш}}^{1,1}\right)}_1 \circ \underset{-\sigma}{\overline{Ш}}^{2,n-2} = 0.$$