Braided diagrams as a unifying tool in homology theory

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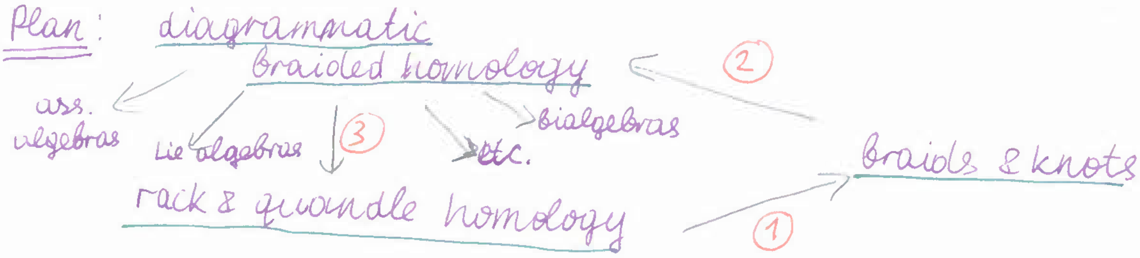
Overview: Diagrammatic calculus
- → categories
- → quantum groups
- → operads
- → categorification
- → algebraic homology ↜ "braided" diagrams

here

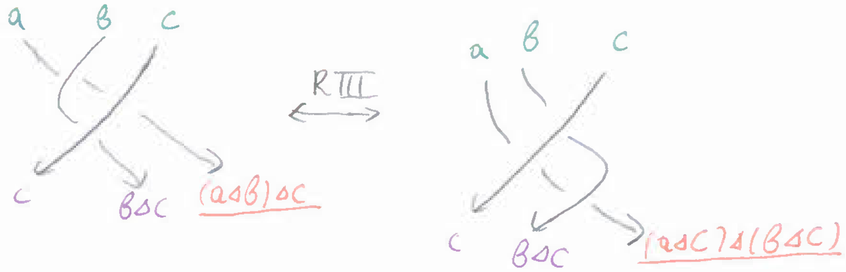
⇒ connections with braid & knot theories

Plan:



① (Co-)homology of self-distributive structures

diagram colorings by $(S, \triangleleft)$



Algebra			Topology
shelf	$(a\triangleleft b)\triangleleft c = (a\triangleleft c)\triangleleft(b\triangleleft c)$ (SD)	⟷ RIII	positive braids
rack	& all $S \to S$, $a \mapsto a\triangleleft b$ are bijective	⟷ & RII:  $= ||$	braids
quandle	& $a\triangleleft a = a$	⟷ & RI:  $= |$	or. knots (& links)
kei	& $(a\triangleleft b)\triangleleft b = a$	⟷ & indep. on orientation	unor. knots

Examples

	Braids	Knots
• ($\text{grp } G$, $a \triangleleft b = \underline{b^{-k} a b^{k}}$) quandle	Artin: $B_n \curvearrowright F_n$	Wirtinger
• ($\text{grp } G$, $a \triangleleft b = \underline{b a^{-1} b}$) kei		
↳ ($\mathbb{Z}_n$, $a \triangleleft b = \underline{2b - a}$)		Fox colorings e.g. $n=3$:  or 
• ($M \in \underset{\Lambda}{\mathbb{Z}[t^{\pm 1}]}\text{Mod}$, $a \triangleleft b = \underline{ta + (1-t)b}$) quandle	Burau: $B_n \to GL_n(\Lambda)$	Alexander
• ($\mathbb{Z}$, $a \triangleleft b = \underline{a+1}$) rack	word length	writhe, linking nbs
• free shelves	Dehornoy order	

Theory: Joyce, Matveev '82

$$\boxed{\text{Knots}/_{K=-K^*} \hookrightarrow \text{Quandles}}$$
← weak universal knot invariant

$$K \longmapsto \underline{FQ(K)}$$
↖ fundamental quandle

$$\longmapsto \langle a, b, c \mid b = a \triangleleft c,\ a = c \triangleleft b,\ c = b \triangleleft a \rangle$$

Practice: $\mathrm{Hom}_{Qu}(FQ(K), \overset{\text{a nice quandle}}{Q}) \simeq \{Q\text{-colorings of a diag. of } K\}$

$\#\swarrow$ a number

Ex.: Fox

$\downarrow$ Boltzmann weights

multi-set

$$w: S \times S \to \underset{\text{ab. grp}}{A} \rightsquigarrow \{\{ W_w(\mathcal{C}) = \sum_{\substack{a \searrow\!\!\swarrow \\ b}} \pm w(a,b) \mid \mathcal{C} \in Col_Q(D) \}\}$$

is a knot invar. iff

(W1) $\underline{w(a,a) = 0}$

(W3) $\underline{w(a \triangleleft c, b \triangleleft c) + w(a,c) = w(a,b) + w(a \triangleleft b, c)}$

General^{ns}:

- color regions by a Q-module (M, $\blacktriangleleft: M \times Q \to M$)
  ↳ $(m \blacktriangleleft a) \blacktriangleleft b = (m \blacktriangleleft b) \blacktriangleleft (a \triangleleft b)$,
  ↳ ($m \mapsto m \blacktriangleleft a$ are bijective)

$$\left.\right\} \quad w: M \times S^{\times(n+1)} \to A \text{ s.t. } \dots$$

- $K^n \hookrightarrow \mathbb{R}^{n+2}$

Self-distributive world

- $(Q, \triangleleft)$ shelf $\rightsquigarrow V = \mathbb{Z}Q$
- $(\widetilde{M}, \blacktriangleleft)$ Q-module $\rightsquigarrow M = \mathbb{Z}\widetilde{M}$

Associative world

- $(V, \cdot)$ ass. algebra
- $(M, \cdot)$ V-module

pre-cubical structure $\begin{cases} d_i^l \ \& \ d_i^r : M \otimes V^{\otimes n} \to M \otimes V^{\otimes (n-1)}, & 1 \leq i \leq n \\ \text{s.t. } d_i^\varepsilon \circ d_j^\zeta = d_{j-1}^\zeta \circ d_i^\varepsilon, & i \leq j, \ \varepsilon, \zeta \in \{l, r\} \end{cases}$ $\Rightarrow$ $\partial^2 = 0$ for $\partial = \alpha \partial^l + \beta \partial^r$, $\partial^\varepsilon = \sum_i (-1)^{i-1} d_i^\varepsilon$

$(m, a_1, \dots, a_n) \xmapsto{d_i^l} \underline{(m \blacktriangleleft a_i, a_1 \triangleleft a_i, \dots, a_{i-1} \triangleleft a_i, a_{i+1}, \dots)}$

$ \xmapsto{d_i^r} \underline{(m, a_1, \dots, a_{i-1}, a_{i+1}, \dots)}$

Th. (Fenn et al., Przytycki et al.): It is a pre-cubical structure.

$m \otimes v_1 \dots \otimes v_n \xmapsto{d_i^l} \dots \otimes v_{i-1} \cdot v_i \otimes \dots$

Th.: — " —.

cubical structure: $\begin{cases} \& \ s_i : M \otimes V^{\otimes n} \to M \otimes V^{\otimes (n+1)}, & 1 \leq i \leq n \\ \text{s.t. } d_j^\varepsilon \circ s_i = \dots \end{cases}$ $\Rightarrow$ $(\sum_i \operatorname{Im} s_i, \partial_*) \hookrightarrow (M \otimes T(V), \partial_*) \twoheadrightarrow \dots$

↙ degeneracies

$$\begin{tikzcd} (M \otimes T(V), \partial_*) \arrow[r, two heads] \arrow[d, squiggly] & \dots \arrow[d, squiggly] \\ H_*(M, V) & H_*^N(M, V) \\ H^*(M, V; A) & H_N^*(M, V; A) \end{tikzcd}$$

$(m, a_1, \dots, a_n) \xmapsto{s_i} \underline{(\dots, a_i, a_i, \dots)}$

Th. (Carter et al.): It is a cubical str. if S is a quandle.

$m \otimes v_1 \otimes \dots \otimes v_n \xmapsto{s_i} \underline{\dots \otimes v_{i-1} \otimes 1 \otimes v_i \otimes \dots}$

Th.: It is a cubical str. if V is unital.

$\boxed{\text{etc.}}$

$\to \partial^l$: 1-term distributive (co-)hom.

$\to \partial^l - \partial^r$: rack (co-)hom.

$\to \partial^l - \partial^r$ & normal^n: quandle (co-)hom.

- 2-cocycles $\Leftrightarrow$ (W1) & (W2) $\rightsquigarrow$ knot invar.
- 2-coboundaries $\rightsquigarrow$ trivial knot invar.
- $w \in H_N^{n+1} \rightsquigarrow$ invar. of $K^n \hookrightarrow \mathbb{R}^{n+2}$

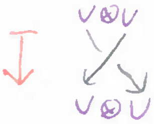
$\to$ (normalized) bar c-x

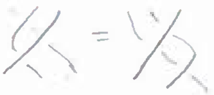
$\to$ Hochschild

? (J. Przytycki)

② Braided (co-)homology: when parallel theories intersect.

Braided vector space: $(V, \sigma : V^{\otimes 2} \circlearrowleft)$ s.t. $\underbrace{\sigma_1 \circ \sigma_2 \circ \sigma_1 = \sigma_2 \circ \sigma_1 \circ \sigma_2}$ (YBE) — Yang-Baxter equation; $\left\{ \begin{array}{l} \sigma_1 = \sigma \otimes Id_V, \\ \sigma_2 = Id_V \otimes \sigma \end{array} \right.$



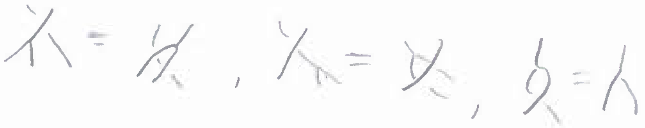


$V^{\otimes 3} \downarrow V^{\otimes 3}$

⚠ We don't need σ^{-1} here.

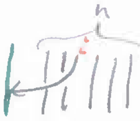
Braided V-module: $(M, M \otimes V \xrightarrow{\rho} M)$ s.t.

Braided co-algebra: $(\underbrace{V, \sigma}_{\text{br. v. sp.}}, \Delta : V \to V \otimes V)$ s.t.



⚠ Cf. knotted trivalent graphs.

Th. (L. 2013): ① $\left\{ \begin{array}{l} (V, \sigma) \\ (M, \rho) \end{array} \right.$ ⇝ pre-simplicial structure on $M \otimes T(V)$

$d_i =$ 

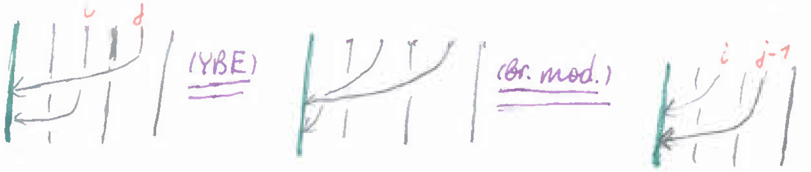
$= \rho \circ \sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_{i-1}$

② & (V, σ, Δ) ⇝ simplicial str.

$s_i =$ 

$= \Delta_i$

Proof: ①



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Remarks:

- two-sided version: $d_i^r =$ 

- functoriality
- works in a pre-additive monoidal category
- genⁿ: colored braids & multi-component braided systems
- a rich theory.

$$\begin{tikzcd}[column sep=small] ③\ \text{SD / associative structures} \arrow[rr, mapsto] \arrow[dr, "\text{classical methods}"'] & & \text{Braided v. sp.} \arrow[dl, "\text{Theorem}"] \\ & \text{(co-)homology theory} \arrow[loop above] & \end{tikzcd}$$

	SD	ass.	Leibniz
σ	$\begin{array}{c}(a,b)\\ \downarrow\\ (b, a\triangleleft b)\end{array}$	$\begin{array}{c}v\otimes w\\ \downarrow\\ 1\otimes v\cdot w\end{array}$	$\begin{array}{c}v\otimes w\\ \downarrow\\ w\otimes v+1\otimes[v,w]\end{array}$
(YBE) $\Leftrightarrow$	(SD)	ass-ty (if $v\cdot 1=v$)	$[v,[w,u]]+[[v,u],w]=[[v,w],u]$ (Lei) (if $[v,1]=[1,v]=0$)
$\exists\sigma^{-1}$?	$\Leftrightarrow$ rack	no	yes
Braided modules	usual	modules	
Δ	$a\mapsto(a,a)$	$v\mapsto 1\otimes v$	$v\mapsto 1\otimes v+v\otimes 1$, $v\in V'$ $1\mapsto 1\otimes 1$ $V=V'\oplus k1$
 $\Leftrightarrow$	$a\triangleleft a=a$	$1\cdot v=v$	$[1,1]=[1,v]+[v,1]=0$ $[V',V']\subseteq V'$
Braided (co)hom. $\supseteq$	• 1-term distr. • rack • quandle	• bar • Hochschild	• Leibniz

$$\begin{tikzcd} \text{Leibniz} \arrow[d, shift left, "{[v,w]=-[w,v]}"] \\ \text{Lie} \arrow[u, hook, shift left] \end{tikzcd} \qquad \begin{tikzcd} T(V) \arrow[d, Rightarrow] & \text{Loday, Cuvier} \\ \Lambda(V), & \text{Chevalley Eilenberg} \end{tikzcd}$$

Other "Braidable" structures

- Bialgebra
- Hopf (bi-)module
- Yetter-Drinfel'd module
- (weak) Poisson algebras
- multi-conjugation quandles $(Q=\sqcup_i G_i, \triangleleft)$ G_i grps

etc. . . .